Multilinear forms and determinants

Last time: A multilinear form

$$A : \overbrace{V \times V \times \cdots \times V}^{k} \to F$$

is a function which takes in k input vectors and gives a scalar such that, in each position, we have

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x} + \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k) =$$
$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x}, \vec{v}_{j+1}, \ldots, \vec{v}_k) + A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, c\vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k) = cA(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

Last time: A multilinear form

$$A : \overbrace{V \times V \times \cdots \times V}^{k} \to F$$

is a function which takes in k input vectors and gives a scalar such that, in each position, we have

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x} + \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k) =$$
$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{x}, \vec{v}_{j+1}, \ldots, \vec{v}_k) + A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{y}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

$$A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, c\vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k) = cA(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{z}, \vec{v}_{j+1}, \ldots, \vec{v}_k).$$

If e_1, e_2, $\ldots$, e_n is a basis for V, then A is determined by the values of $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$. This is n^k values.

A multilinear form is alternating if $A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = 0$ whenever two of the $\vec{v}$'s are equal. This implies that switching any $\vec{v}_i$ and $\vec{v}_j$ switches the sign of A.

A multilinear form is alternating if $A(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) = 0$ whenever two of the $\vec{v}$'s are equal. This implies that switching any $\vec{v}_i$ and $\vec{v}_j$ switches the sign of A.

So, if any two of i_1, i_2, …, i_k are equal, then $A(e_{i_1}, e_{i_2}, \dots, e_{i_k}) = 0$.

A multilinear form is alternating if $A(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = 0$ whenever two of the $\vec{v}$'s are equal. This implies that switching any $\vec{v}_i$ and $\vec{v}_j$ switches the sign of A.

So, if any two of i_1, i_2, …, i_k are equal, then $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) = 0$.

And, if i_1, i_2, …, i_k are distinct, then the $k!$ values of A where we order the inputs in different ways all differ up to sign. So A is determined by the values $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ where $i_1 < i_2 < \cdots < i_k$.

For example, if $A(e_1, e_2, e_3) = a$, then we have

$$A(e_1, e_2, e_3) = A(e_2, e_3, e_1) = A(e_3, e_1, e_2) = a \text{ and}$$

$$A(e_1, e_3, e_2) = A(e_3, e_2, e_1) = A(e_2, e_1, e_3) = -a.$$

But it isn't clear that, for general k, there is a consistent way to choose the signs.

Okay, clearing up the sign issue. Suppose we know that $A(e_1, e_2, e_3, e_4, e_5) = 7$. What should $A(e_3, e_5, e_1, e_2, e_4)$ be?

Okay, clearing up the sign issue. Suppose we know that $A(e_1, e_2, e_3, e_4, e_5) = 7$. What should $A(e_3, e_5, e_1, e_2, e_4)$ be?

$$\begin{aligned} A(e_3, e_5, e_1, e_2, e_4) &= -A(e_3, e_4, e_1, e_2, e_5) \\ &= A(e_3, e_2, e_1, e_4, e_5) \\ &= -A(e_1, e_2, e_3, e_4, e_5) = -7 \end{aligned} .$$

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

What we need to show is that there is a way to assign a sign to every permutation of $\{1, 2, 3, \ldots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{llll} (1,2,3) \rightsquigarrow & 1 & (1,3,2) \rightsquigarrow & -1 \\ (2,1,3) \rightsquigarrow & -1 & (2,3,1) \rightsquigarrow & 1 \\ (3,1,2) \rightsquigarrow & 1 & (3,2,1) \rightsquigarrow & -1 \end{array}$$

What we need to show is that there is a way to assign a sign to every permutation of $\{1, 2, 3, \ldots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{rrrr} (1,2,3) \rightsquigarrow & 1 & (1,3,2) \rightsquigarrow & -1 \\ (2,1,3) \rightsquigarrow & -1 & (2,3,1) \rightsquigarrow & 1 \\ (3,1,2) \rightsquigarrow & 1 & (3,2,1) \rightsquigarrow & -1 \end{array}$$

Here is the rule: The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

We need to show that, if we switch $\sigma(i)$, $\sigma(j)$, then this changes sign.

Of course, we switch the contribution from (i, j). Also, if $i < h < j$ and $\sigma(h)$ is between $\sigma(i)$ and $\sigma(j)$, the contribution to the exponent from (i, h) and (h, j) will switch by 2, so the sign will not change.

The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

We need to show that, if we switch $\sigma(i)$, $\sigma(j)$, then this changes sign.

Of course, we switch the contribution from (i, j). Also, if $i < h < j$ and $\sigma(h)$ is between $\sigma(i)$ and $\sigma(j)$, the contribution to the exponent from (i, h) and (h, j) will switch by 2, so the sign will not change.

Now we know that there is a well defined sign for each permutation. So, given the values $A(e_{i_1}, e_{i_2}, \ldots, e_{i_k})$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we can describe A on all k-tuples of basis vectors as follows:

If two of j_1, j_2, …, j_k are equal, then $A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = 0$.

If all of the j_1, j_2, …, j_k are distinct, and $j_{\sigma(1)} < j_{\sigma(2)} < \cdots < j_{\sigma(k)}$, then

$$A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = \mathtt{sign}(\sigma) A(e_{j_{\sigma(1)}}, e_{j_{\sigma(2)}}, \ldots, e_{j_{\sigma(k)}}).$$

Thus, if $\dim V = n$, the space of alternating multilinear forms of k vectors has dimension $\binom{n}{k}$.

Corollary: If $k > n$, the only alternating multilinear form $V^k \to F$ is 0.

Corollary: If $k = n$, there a one dimensional space of alternating multilinear form $V^k \to F$.

Determinants

Let $n = \dim V$. We have seen that there is a one dimensional space of alternating multilinear forms $\overbrace{V \times V \times \cdots \times V}^{n} \to F$. Our goal now will be to understand this concretely.

Let $n = \dim V$. We have seen that there is a one dimensional space of alternating multilinear forms $\overbrace{V \times V \times \cdots \times V}^{n} \to F$. Our goal now will be to understand this concretely.

Let ω be a nonzero alternating multilinear form of n vectors. Let $T : V \to V$ be a linear transformation. Then the function

$$\omega(T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n)$$

is also an alternating multilinear function.

Let $n = \dim V$. We have seen that there is a one dimensional space of alternating multilinear forms $\overbrace{V \times V \times \cdots \times V}^{n} \to F$. Our goal now will be to understand this concretely.

Let ω be a nonzero alternating multilinear form of n vectors. Let $T : V \to V$ be a linear transformation. Then the function

$$\omega(T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n)$$

is also an alternating multilinear function. So it is a scalar multiple of ω. We call this scalar $\det(T)$:

$$\omega(T\vec{v}_1, T\vec{v}_2, \ldots, T\vec{v}_n) = \det(T)\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n).$$

Let's see how we compute det from this definition.

We'll start with $n = 2$. Let T have matrix $\left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right]$.

Let's see how we compute det from this definition.

We'll start with $n = 2$. Let T have matrix $\left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right]$.

$$\omega(Te_1, Te_2) = \omega(T_{11}e_1 + T_{21}e_2, T_{21}e_1 + T_{22}e_2) =$$

$$T_{11}T_{21}\omega(e_1, e_1) + T_{11}T_{22}\omega(e_1, e_2) + T_{21}T_{12}\omega(e_2, e_1) + T_{21}T_{22}\omega(e_2, e_2)$$

$$= 0 + T_{11}T_{22}\omega(e_1, e_2) - T_{21}T_{12}\omega(e_1, e_2) + 0 = (T_{11}T_{22} - T_{21}T_{12})\omega(e_1, e_2).$$

Let's see how we compute det from this definition.

We'll start with $n = 2$. Let T have matrix $\left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right]$.

$$\omega(Te_1, Te_2) = \omega(T_{11}e_1 + T_{21}e_2, T_{12}e_1 + T_{22}e_2) =$$

$$T_{11}T_{12}\omega(e_1, e_1) + T_{11}T_{22}\omega(e_1, e_2) + T_{21}T_{12}\omega(e_2, e_1) + T_{21}T_{22}\omega(e_2, e_2)$$

$$= 0 + T_{11}T_{22}\omega(e_1, e_2) - T_{21}T_{12}\omega(e_1, e_2) + 0 = (T_{11}T_{22} - T_{21}T_{12})\omega(e_1, e_2).$$

So

$$\det \left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right] = (T_{11}T_{22} - T_{21}T_{12}).$$

Similarly, for $n = 3$, expanding $\omega(Te_1, Te_2, Te_3)$ gives 3^3 terms:

$$T_{11}T_{12}T_{13}\omega(e_1, e_1, e_1) + T_{11}T_{12}T_{23}\omega(e_1, e_1, e_2) + \cdots$$

All but 3! of them are 0. The nonzero ones give

$$T_{11}T_{22}T_{33}\omega(e_1, e_2, e_3) + T_{11}T_{32}T_{23}\omega(e_1, e_3, e_2) + \cdots =$$

$$(T_{11}T_{22}T_{33} - T_{11}T_{32}T_{23} - T_{21}T_{12}T_{33} + T_{21}T_{32}T_{13} + T_{31}T_{12}T_{23} - T_{31}T_{22}T_{13}) \cdot \omega(e_1, e_2, e_3).$$

Similarly, for $n = 3$, expanding $\omega(Te_1, Te_2, Te_3)$ gives 3^3 terms:

$$T_{11}T_{12}T_{13}\omega(e_1, e_1, e_1) + T_{11}T_{12}T_{23}\omega(e_1, e_1, e_2) + \cdots$$

All but 3! of them are 0. The nonzero ones give

$$T_{11}T_{22}T_{33}\omega(e_1, e_2, e_3) + T_{11}T_{32}T_{23}\omega(e_1, e_3, e_2) + \cdots =$$

$$(T_{11}T_{22}T_{33} - T_{11}T_{32}T_{23} - T_{21}T_{12}T_{33} + T_{21}T_{32}T_{13} + T_{31}T_{12}T_{23} - T_{31}T_{22}T_{13}) \cdot \omega(e_1, e_2, e_3).$$

In general, we get

$$\sum_{\sigma \text{ a permutation of } \{1,2,3,\ldots,n\}} \texttt{sign}(\sigma) T_{\sigma(1)1} T_{\sigma(2)2} \cdots T_{\sigma(n)n}.$$

We started by saying that there is a one dimensional space of alternating multilinear forms $\overbrace{V \times V \times \cdots \times V}^{n} \to F$.

Concretely, all such forms are a scalar multiple of

$$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \longrightarrow \det(\vec{v}_1 \;\; \vec{v}_2 \;\; \cdots \;\; \vec{v}_n).$$

Properties of determinant

Multilinearity gives the standard column properties of determinant:

- If we switch two columns, we switch the sign of the determinant.
- If we rescale a column, the determinant rescales.
- If we add a multiple of one column to another, the determinant is unchanged.

These properties also hold for rows, since $\det(A) = \det(A^T)$.

We have $\det(T) = 0$ if and only if T is not invertible. In other words, if ω is a nonzero alternating form, then $\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = 0$ if and only if $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$ are linearly dependent.

We have $\det(T) = 0$ if and only if T is not invertible. In other words, if ω is a nonzero alternating form, then $\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = 0$ if and only if $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$ are linearly dependent.

Proof: Suppose that we had a linear dependency $\vec{v}_j = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1}$. Then

$$\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1}, \cdots \vec{v}_n)$$

$$= \sum_{k=1}^{j-1} c_k \omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}, \vec{v}_k, \ldots, \vec{v}_n) = 0 + 0 + \cdots + 0.$$

We have $\det(T) = 0$ if and only if T is not invertible. In other words, if ω is a nonzero alternating form, then $\omega(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$ if and only if $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ are linearly dependent.

Proof: Suppose that we had a linear dependency $\vec{v}_j = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{j-1}\vec{v}_{j-1}$. Then

$$\omega(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = \omega(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{j-1}, c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{j-1}\vec{v}_{j-1}, \cdots \vec{v}_n)$$

$$= \sum_{k=1}^{j-1} c_k \omega(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{j-1}, \vec{v}_k, \dots, \vec{v}_n) = 0 + 0 + \dots + 0.$$

In the other direction, suppose for the sake of contradiction that $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ is a basis of V, and yet $\omega(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = 0$. Since ω is determined by its value on a basis, we deduce that ω is identically zero; contradiction. $\square$

The most exciting property of determinant is that it is multiplicative:

$$\det(AB) = \det(A)\det(B).$$

Let's prove this!

The most exciting property of determinant is that it is multiplicative:

$$\det(AB) = \det(A)\det(B).$$

Let's prove this!

$$\begin{aligned} \omega(AB\vec{v}_1, AB\vec{v}_2, \ldots, AB\vec{v}_n) &= \det(A)\omega(B\vec{v}_1, B\vec{v}_2, \ldots, B\vec{v}_n) \\ &= \det(A)\det(B)\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \end{aligned}.$$

The most exciting property of determinant is that it is multiplicative:

$$\det(AB) = \det(A)\det(B).$$

Let's prove this!

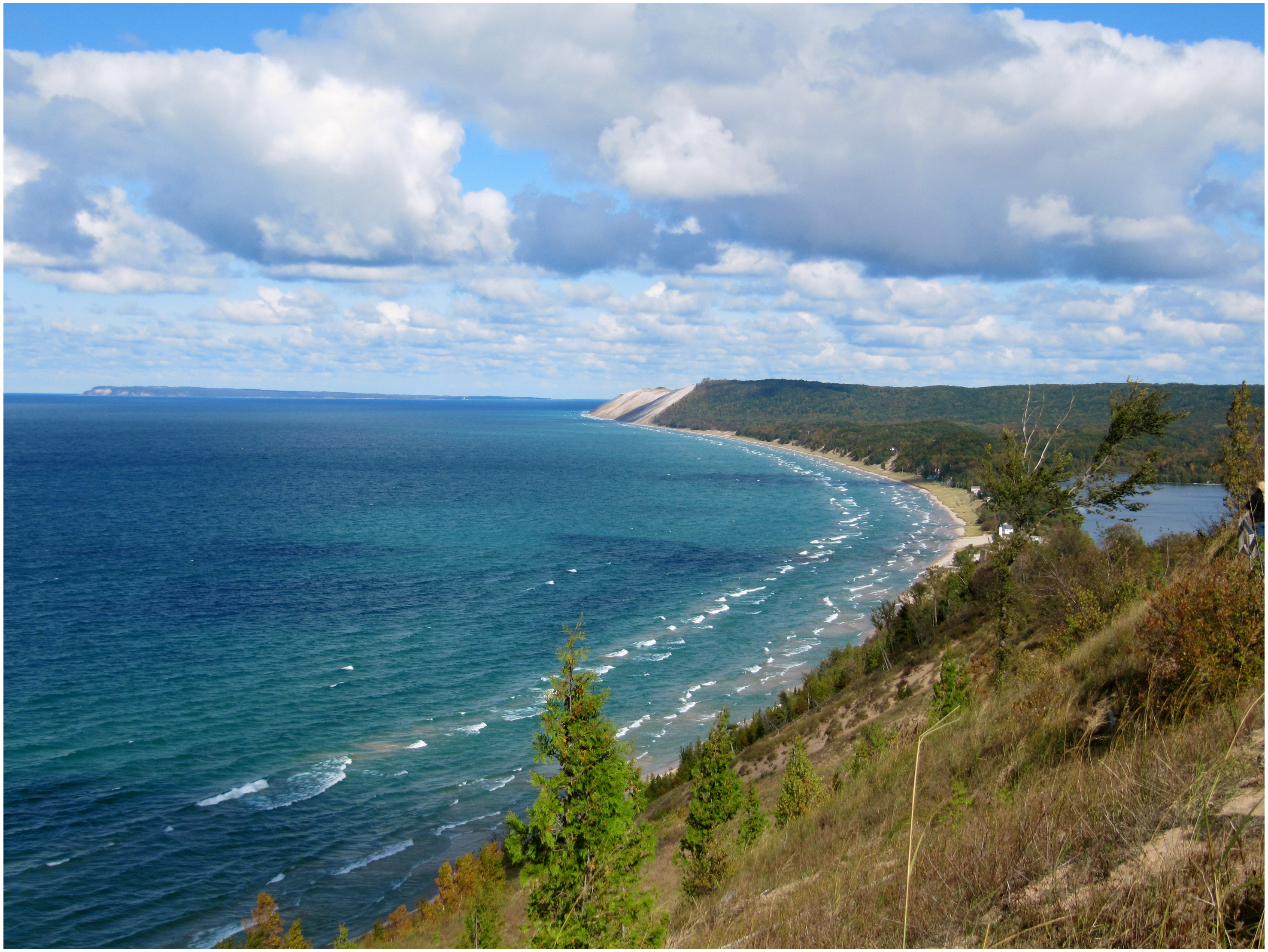
$$\begin{aligned} \omega(AB\vec{v}_1, AB\vec{v}_2, \ldots, AB\vec{v}_n) &= \det(A)\omega(B\vec{v}_1, B\vec{v}_2, \ldots, B\vec{v}_n) \\ &= \det(A)\det(B)\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) \end{aligned}.$$

But also

$$\omega(AB\vec{v}_1, AB\vec{v}_2, \ldots, AB\vec{v}_n) = \det(AB)\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n).$$

So

$$\det(AB) = \det(A)\det(B). \qquad \square$$



Enjoy your break! More about determinants when we return!