Basic computations

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 2 & 6 & 2 & 2 & 4 & 2 \\ 1 & 3 & 2 & 3 & 5 & -1 \\ -1 & -3 & -1 & -1 & -2 & 1 \\ 1 & 3 & 1 & 1 & 2 & -2 \end{bmatrix}.$$

The row reduction (reduced row-echelon form) of A is

$$\begin{bmatrix} 1 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) Compute a basis for the image of A.
(2) Compute a basis for the kernel of A.

Problem 2. Find a basis for the vector space of real polynomials of degree ≤ 3 obeying $f(1) = f(-1) = 0$.

Problem 3. Let $\vec{x} = \left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right]$, $\vec{y} = \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$ and $\vec{z} = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$. What are the coordinates of the vector $\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$ in the basis $\vec{x}$, $\vec{y}$, $\vec{z}$?

Basic proofs

Problem 4. Let V be a vector space. Prove, directly from the axioms of a vector space, that $-(-\vec{v}) = \vec{v}$ for all $\vec{v} \in V$.

Problem 5. Let V be a vector space over a field F. Let $a \in F$ and let $\vec{v} \in V$ and suppose that $a\vec{v} = \vec{0}$. Show that either $a = 0$ or $\vec{v} = \vec{0}$ (or both). In addition to the axioms of a vector space, you may use that $c\vec{0} = \vec{0}$ and $0\vec{x} = \vec{0}$.

Problem 6. Let V be a vector space and let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be linearly independent vectors in V. Let $\vec{w}$ be an additional vector. Show that $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{w})$ is linearly dependent if and only if $\vec{w} \in \mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

Problem 7. Let F be a field and let F^∞ be the vector space of infinite sequences $(a_1, a_2, a_3, \ldots,)$ of elements of F. Let $\vec{e}_i$ be the element $(0, 0, 0, \ldots, 1, 0, \ldots)$ of F^∞, whose single one is in the i-th position; let $\vec{f}$ be the vector $(1, 1, 1, 1, \ldots)$.

(1) Show that the infinite list of vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, …, $\vec{f}$ is linearly independent.
(2) Give (and prove your answer correct) a vector $\vec{g}$ which is not in the span of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, …, $\vec{f}$.

Problem 8. Let V be a finite dimensional vector space and let X and Y be subspaces with $X \cap Y = \{0\}$. Show that $\dim V \geq \dim X + \dim Y$.

Problem 9. Let V be a vector space over a field F and let X and Y be subspaces of V. Suppose that $\vec{w}_1$, $\vec{w}_2$, $\ldots$, $\vec{w}_a$ is a basis of $X \cap Y$, that $\vec{w}_1$, $\vec{w}_2$, $\ldots$, $\vec{w}_a$, $\vec{x}_1$, $\vec{x}_2$, $\ldots$, $\vec{x}_b$ is a basis of X and that $\vec{w}_1$, $\vec{w}_2$, $\ldots$, $\vec{w}_a$, $\vec{y}_1$, $\vec{y}_2$, $\ldots$, $\vec{y}_c$ is a basis of Y. Show that the list of vectors $\vec{w}_1$, $\vec{w}_2$, $\ldots$, $\vec{w}_a$, $\vec{x}_1$, $\vec{x}_2$, $\ldots$, $\vec{x}_b$, $\vec{y}_1$, $\vec{y}_2$, $\ldots$, $\vec{y}_c$ is linearly independent.

Problem 10. Let V be a vector space and let $T : V \to V$ be a linear transformation obeying $T^2 = T$. Let $K = \mathrm{Ker}(T)$ and let $I = \mathrm{Image}(T)$. Show that $V = K \oplus I$.

Problem 11. Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. Show that $\dim \mathrm{Ker}(T^2) \leq 2 \dim \mathrm{Ker}(T)$.