Spaces of linear maps, dual spaces, transpose

Overview of today: We are going to introduce a notion of the "dual" to a vector space. The difference between the original vectors and the dual vectors can be thought of like the difference between column vectors and row vectors. Keeping track of the difference between vectors and dual vectors may seem fiddly at first, but I hope it will eventually be clarifying.

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Again, this is mn basis vectors. $\square$

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If V is finite dimensional, with basis e_1, e_2, $\ldots$, e_n, then e_i^* is a basis of V. So $\dim V = \dim V^*$ in the finite dimensional case. Think of elements of V as column vectors and elements of V^* as row vectors.

We will avoid infinite dimensional issues in this lecture.

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Notation check: What is $F(e_1)$? What is $F(e_2)$?

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So the matrix of F is the transpose matrix, $\left[\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix}\right]$.

In general, suppose we have bases e_1, e_2, $\ldots$, e_m of V and f_1, f_2, $\ldots$, f_n of W. Let e_i^* and f_j^* be the dual bases of V^* and W^*. Then, if A is the matrix of $F : V \to W$, then A^T is the matrix of $F^* : W^* \to V^*$.

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You may recall the identity $(AB)^T = B^T A^T$. We can prove this without coordinates, the claim is that $(FG)^* = G^* F^*$, for $G : U \to V$ and $F : V \to W$.

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If you have ever felt unsatisifed by transpose, here is where it comes from.

Theorem: If $F : V \to W$ is surjective, then $F^* : W^* \to V^*$ is injective.

Reminder: "Surjective" means "image is everything"; "injective" means "kernel is zero".

In finite dimensions, this says "if a matrix is surjective, then its transpose is injective".

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But our assumption is that Image(F) is everything. So this shows that w^* is 0 for every input, and $w^* = 0$, as desired. $\square$

What about the reverse direction? If $F : V \to W$ is injective, is $W^* \to V^*$ surjective?

Another way to put this is: If v^* is any linear functional $V \to F$, and V embeds as a subspace of W, can we extend v^* to a linear functional on W?

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Theorem: If V and W are finite dimensional, and $F : V \to W$ is injective, then $W^* \to V^*$ is surjective.

Proof: Let $\vec{v}_1$,$\vec{v}_2$, $\ldots$, $\vec{v}_k$ be a basis of V, so that $F(\vec{v}_1)$, $F(\vec{v}_2)$, $\ldots$, $F(\vec{v}_k)$ is a basis of $F(V)$. Then extend $F(\vec{v}_1)$, $F(\vec{v}_2)$, $\ldots$, $F(\vec{v}_k)$ to a basis $F(\vec{v}_1)$, $F(\vec{v}_2)$, $\ldots$, $F(\vec{v}_k)$, $\vec{w}_1$, $\vec{w}_2$, $\ldots$,$\vec{w}_{n-k}$ of W. Put $U = \text{Span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{n-k})$. Then $W = U \oplus F(V)$. So every vector $\vec{w} \in W$ can be written uniquely as $\vec{u} + F(\vec{v})$. Then define $w^*(\vec{w})$ to be $v^*(\vec{v})$. $\square$

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The infinite dimensional case is not obvious! For example, consider $\mathbb{R}^\infty_{\text{finite}} \subset \mathbb{R}^\infty$. We have a linear functional $(a_1, a_2, \ldots) \mapsto \sum a_i$ on $\mathbb{R}^\infty_{\text{finite}}$. If we can extend this to $\mathbb{R}^\infty$, this means that we can define a notion of sum for every sequence, no matter how divergent. More later ...