Theorems about invertible matrices

Remember last time: A matrix is in ***row reduced echelon form if:***

- Either row is either all 0's, or else its first nonzero entry is a 1. This 1 is called a ***pivot***.
- In a column which contains a pivot, called a ***pivot column***, all the other entries are 0.
- The nonzero rows are at the top of the matrix; they are ordered so that the pivots go from left to right as we go down the rows.

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Theorem: For any matrix A, there is an invertible matrix U such that UA is in row reduced echelon form.

Theorem: Let A be an $m \times n$ matrix. The following are equivalent:

1. Every column of $\mathrm{rref}(A)$ is a pivot column.
2. There is an $n \times m$ matrix B with $BA = \mathrm{Id}_n$.
3. $\mathrm{Ker}(A) = \{\vec{0}\}$.
4. For all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, if $A\vec{x} = A\vec{y}$ then $\vec{x} = \vec{y}$. (Vocabulary: A is ***injective***.)

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Note: If these conditions hold, then $m \geq n$.

Proofs: (1) $\implies$ (2): If every column is a pivot column, the row reduced form of A must be

$$R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So there is some invertible U with $UA = R$.

Now, $QR = \mathrm{Id}_n$ where

$$Q = \begin{bmatrix} 1 & & & & 0 & 0 & 0 \\ & 1 & & & 0 & 0 & 0 \\ & & 1 & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \end{bmatrix}.$$

So $(QU^{-1})(UR) = \mathrm{Id}_n$. Take $B = QU^{-1}$.

(2) $\implies$ (3): Suppose that $A\vec{x} = \vec{0}$. Then $BA\vec{x} = B\vec{0} = \vec{0}$ so we deduce that $\vec{x} = \vec{0}$.

(3) $\implies$ (4): Suppose that $A\vec{x} = A\vec{y}$. Then $A(\vec{x} - \vec{y}) = 0$, so $\vec{x} - \vec{y} = \vec{0}$ and we deduce that $\vec{x} = \vec{y}$.

$\texttt{NOT}(1) \implies \texttt{NOT}(4)$: Let R be the row reduced form of A, and let the k-th column be a free column. Let the pivot columns be p_1, p_2, $\ldots$, p_r.

$$R = \begin{bmatrix} 1 & 0 & R_{1k} & * & 0 \\ 0 & 1 & R_{2k} & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $Re_k = R \sum_j R_{jk} e_{p_j}$. In the example,

$$\begin{bmatrix} 1 & 0 & R_{1k} & * & 0 \\ 0 & 1 & R_{2k} & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & R_{1k} & * & 0 \\ 0 & 1 & R_{2k} & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{13} \\ R_{23} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then $URe_k = UR \sum_j R_{jk} e_{p_j}$, where $A = UR$. $\square$

Theorem: Let A be an $m \times n$ matrix. The following are equivalent:

1. Every row of $\mathrm{rref}(A)$ is a pivot row.
2. There is an $n \times m$ matrix C with $AC = \mathrm{Id}_m$.
3. $\mathrm{Image}(A) = \mathbb{R}^m$. (Vocabulary: A is **surjective**).

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Note, if these conditions happen, then $m \leq n$.

Proofs: (1) $\implies$ (2): Let R be the row reduced form, and let $A = UR$. So R looks like

$$R = \left[\begin{smallmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{smallmatrix}\right].$$

Then $RQ = \mathrm{Id}$, where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And then $(UR)(QU^{-1}) = \mathrm{Id}$ as well.

(2) $\implies$ (3): Suppose that $AC = \mathrm{Id}_m$. Let $\vec{y}$ be in $\mathbb{R}^m$. Then we claim that $\vec{x} = C\vec{y}$ is a solution to $A\vec{x} = \vec{y}$. Indeed, $AC\vec{y} = \mathrm{Id}\vec{y} = \vec{y}$.

$\texttt{NOT}(1) \implies \texttt{NOT}(3)$. Let R be the row reduced form, and let $A = UR$. Suppose that the bottom row of R is not a pivot row. Then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the image of R. So $U \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the image of A.

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1. Every column of $\mathrm{rref}(A)$ is a pivot column.
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3. $\mathrm{Ker}(A) = \{\vec{0}\}$.
4. A is injective.

If these conditions hold, then $m \geq n$.

Theorem: Let A be an $m \times n$ matrix. The following are equivalent:

1. Every row of $\mathrm{rref}(A)$ is a pivot row.
2. There is an $n \times m$ matrix C with $AC = \mathrm{Id}_m$.
3. A is surjective,

If these conditions hold, then $m \leq n$.

Corollary: If A has an inverse, then $m = n$.

Corollary: If A is injective and surjective, then $m = n$.

Theorem Let $m = n$ and let A be an $m \times n$ matrix. Then the following are equivalent:

1. Every column of $\mathrm{rref}(A)$ is a pivot column.
2. There is an $n \times m$ matrix B with $BA = \mathrm{Id}_n$.
3. $\mathrm{Ker}(A) = \{\vec{0}\}$.
4. A is injective.
5. Every row of $\mathrm{rref}(A)$ is a pivot row.
6. There is an $n \times m$ matrix C with $AC = \mathrm{Id}_m$.
7. A is surjective,

Theorem Let $m = n$ and let A be an $m \times n$ matrix. Then the following are equivalent:

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Proof: We already did the equivalence of (1) – (4), and of (5) – (7). For square matrices, (1) and (5) are obviously equivalent.