Fields, vector spaces, subspaces, linear operators

We have been dealing with three kinds of objects: scalars, vectors and matrices.

We are going to consider more abstract versions of each of these.

- The notion of "a field" will replace $\mathbb{R}$. The elements of the field will be our new version of scalars.
- The notion of "a vector space" will replace $\mathbb{R}^n$. Elements of the vector space will be our new version of vectors.
- The notion of a linear operator with replace a matrix.

Fields

A ***field*** is a set F with two operations $+$ and $\cdot$ and two special elements called 0 and 1, obeying:

$0+x=x+0$	$=x$	$x\cdot 1=1\cdot x$	$=x$	Identity
$x+y$	$=y+x$	$x\cdot y$	$=y\cdot x$	Commutativity
$x+(y+z)$	$=(x+y)+z$	$x\cdot(y\cdot z)$	$=(x\cdot y)\cdot z$	Associativity
$x\cdot(y+z)$	$=x\cdot y+x\cdot z$	$(x+y)\cdot z$	$=x\cdot z+y\cdot z$	Distributivity

For all $x\in F$, there is an element $-x$ such that

$$x+(-x)=(-x)+x=0. \qquad \text{Additive Inverse}$$

If x is a nonzero element of F, there is an element x^{-1} such that

$$x\cdot x^{-1}=x^{-1}\cdot x=1 \qquad \text{Multiplicative Inverse.}$$

$$\text{and } 0\neq 1 \qquad \text{Nontriviality.}$$

Examples of fields: $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers).

More exotically, for any prime p, the field $\mathbb{F}_p$ is integers modulo p.

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

.

In general, all the algebraic identities that you are used to in the real numbers still work in any field. There are ways to make this precise, but let's do some examples instead.

Can you prove:

$$(x + y) \cdot (x + y) = \big(x \cdot x + (1 + 1) \cdot (x \cdot y)\big) + y \cdot y \quad ?$$

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Can you prove:

$$(x+y)\cdot(x+y) = \big(x\cdot x + (1+1)\cdot(x\cdot y)\big) + y\cdot y \quad ?$$

$$\begin{aligned}
& (x+y)\cdot(x+y) & \\
= \;& x\cdot(x+y) + y\cdot(x+y) & \text{Distributive} \\
= \;& (x\cdot x + x\cdot y) + (y\cdot x + y\cdot y) & \text{Distributive (twice)} \\
= \;& \big(x\cdot x + (x\cdot y + y\cdot x)\big) + y\cdot y & \text{Associative (twice)} \\
= \;& \big(x\cdot x + (x\cdot y + x\cdot y)\big) + y\cdot y & \text{Commutative} \\
= \;& \big(x\cdot x + (1\cdot(x\cdot y) + 1\cdot(x\cdot y))\big) + y\cdot y & \text{Mult. Identity (twice)} \\
= \;& \big(x\cdot x + (1+1)\cdot(x\cdot y)\big) + y\cdot y & \text{Distributive}
\end{aligned}$$

This one is harder:

$$x \cdot 0 = 0.$$

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Here is one way:

$$\begin{aligned}
0 + 1 &= 1 && \text{Add. Identity} \\
x \cdot (0 + 1) &= x \cdot 1 && \\
x \cdot 0 + x \cdot 1 &= x \cdot 1 && \text{Distributive} \\
x \cdot 0 + x &= x && \text{Mult Identity} \\
(x \cdot 0 + x) + (-x) &= x + (-x) && \\
x \cdot 0 + (x + (-x)) &= x + (-x) && \text{Associative} \\
x \cdot 0 + 0 &= 0 && \text{Add. Inverse (twice)} \\
x \cdot 0 &= 0 && \text{Add. Identity}
\end{aligned}$$

One more very useful one:

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We have

$$x \cdot (-1) + x = x \cdot (-1) + x \cdot 1 = x((-1) + 1) = x \cdot 0 = 0.$$

So

$$(x \cdot (-1) + x) + (-x) = 0 + (-x).$$

The left hand side is

$$(x \cdot (-1) + x) + (-x) = x \cdot (-1) + (x + (-x)) = x \cdot (-1) + 0 = x \cdot (-1).$$

The right hand side is $-x$. $\square$

Vector spaces

Let F be a field. An ***F-vector space*** is a set V with:

- An operation called $+$ which takes two elements of V and makes a new element of V. (Vector addition)
- An operation called $\cdot$ which takes an elements of V and an element of F and makes a new element of V. (Scalar multiplication)
- A special element called $\vec{0}$.

Let F be a field. An ***F-vector space*** is a set V with $+$, $\cdot$, $\vec{0}$. They obey the field axioms plus:

$\vec{v} + \vec{0} = \vec{0} + \vec{v}$	$= \vec{v}$	$1 \cdot \vec{v}$	$= \vec{v}$	Identity
$\vec{v} + \vec{w}$	$= \vec{w} + \vec{v}$			Commutativity
$\vec{u} + (\vec{v} + \vec{w})$	$= (\vec{u} + \vec{v}) + \vec{w}$	$a \cdot (b \cdot \vec{v})$	$= (a \cdot b) \cdot \vec{v}$	Associativity
$a \cdot (\vec{v} + \vec{w})$	$= a \cdot \vec{v} + a \cdot \vec{w}$	$(a + b) \cdot \vec{v}$	$= a \cdot \vec{v} + b \cdot \vec{v}$	Distributivity

For all $\vec{v} \in V$, there is an element $-\vec{v}$ such that

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$a\cdot(\vec{v}+\vec{w})$	$=a\cdot\vec{v}+a\cdot\vec{w}$	$(a+b)\cdot\vec{v}$	$=a\cdot\vec{v}+b\cdot\vec{v}$	Distributivity

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As before, we can deduce:

$$0\cdot\vec{v}=\vec{0} \qquad a\cdot\vec{0}=\vec{0} \qquad (-1)\vec{v}=-\vec{v}.$$

The obvious example of a vector space is F^n: The list of n-tuples $(x_1, x_2, \ldots, x_n)$ of elements of F, with the standard vector addition and scalar multiplication.

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More generally, if X is any set, then F^X is the set of all functions $X \to F$. We make this into a vector space by

$$(f + g)(x) = f(x) + g(x) \qquad (af)(x) = af(x).$$

So F^n is $F^{\{1,2,\ldots,n\}}$.

Let V be a vector space over a field F. A ***subspace*** of V is a subset L of V such that:

- If $\vec{v}$ and $\vec{w}$ are in L then $\vec{v} + \vec{w}$ is in L.
- If $\vec{v}$ is in L and a is in G then $a\vec{v}$ is in L.

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We can also look at subspaces of F^X for infinite X. For example, consider the vector space $\mathbb{R}^\mathbb{R}$ of all real valued functions on the real line. Then {continuous functions}, or {smooth functions}, or {bounded functions}, all form subspaces of $\mathbb{R}^\mathbb{R}$.

Linear Transformations

Let V and W be two F-vector spaces. A ***linear transformation*** is a map $T : V \to W$ obeying

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) \qquad T(a\vec{v}) = aT(\vec{v}).$$

It is easy to check that

$$T(\vec{0}) = \vec{0} \qquad T(-\vec{v}) = -T(\vec{v}).$$

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The obvious example is $V = F^n$, $W = F^m$ and T is an $m \times n$ matrix. We will soon see that all linear transformations $F^n \to F^m$ come from matrices.

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We can also think of linear transformations of infinite dimensional vector spaces. For example $f \mapsto f(0)$ and $f \mapsto f(1) + f(2) + f(3)$ are both linear transformations $\mathbb{R}^{\mathbb{R}} \to \mathbb{R}$.

The map $f \mapsto \frac{df}{dx}$ is a map from the subspaces {differentiable functions} to the subspace {continuous functions}.

The map $f \mapsto \int_0^1 f(x)dx$ is a map from the subspace {continuous functions} to $\mathbb{R}$.