# Linear independence, spanning sets, bases, dimension

## Before we start: Poll questions

• I'm a little confused on Theorem 6 in 1.4. The theorem itself makes sense, I'm just a bit confused on the wording within the proof. Going over it in lecture might help me to understand.

***Theorem 6.*** *If* A *is an* m × n *matrix and* m < n, *then the homogeneous system of linear equations* AX = 0 *has a non-trivial solution.*

*Proof*. Let $R$ be a row-reduced echelon matrix which is row-equivalent to $A$. Then the systems $AX = 0$ and $RX = 0$ have the same solutions by Theorem 3. If $r$ is the number of non-zero rows in $R$, then certainly $r \leq m$, and since $m < n$, we have $r < n$. It follows immediately from our remarks above that $AX = 0$ has a non-trivial solution. ∎

## Before we start: Poll questions

- If every injective and surjective matrix is invertible, is every invertible matrix injective and surjective?

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Suppose that we have a vector space $V$ over a field $F$. We'd like to treat $V$ like $F^n$. What do we need in order to do so?

We need to find some vectors in $V$ which we can treat like the vectors $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. These vectors, call them $\vec{u}_1$, $\vec{u}_2$, $\ldots$, $\vec{u}_n$, should have two properties:

**Spanning** Every vector $\vec{v}$ in $V$ should be expressible as $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$ for some scalars $a_1$, $a_2$, $\ldots$, $a_n$ in $F$.

**Linearly Independent** No vector in $V$ should be expressible as $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$ in two different ways.

## Example

Let $L$ be the plane $x + y + z = 0$ in $\mathbb{R}^3$.

• If we take $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, then many vectors in $L$ are not multiples of $\vec{u}_1$. **Linearly independent, but not spanning.**

• If we take $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $\vec{u}_3 = \vec{u}_1 + \vec{u}_2$, so vectors can be expressed in more than one way. **Spanning, but not linearly independent.**

• If $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, then every vector in $L$ can be written in exactly one way as a linear combination of $\vec{u}_1$ and $\vec{u}_2$. Namely, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (x + y) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. **Both spanning and linearly independent**

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A linearly independent set is “not too big”. A spanning set is “not too small”. A list of vectors which is both linearly independent and spanning is called a ***basis***.

## More on linear independence

Linear independence can be defined in a less intuitive but more useful way:

**Theorem:** Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ be a list of vectors in a vector space $V$. Then the following conditions are equivalent:

1. For every vector $\vec{v}$ in $V$, there is **at most** one way to write $\vec{v}$ as $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$ for some scalars $a_1, a_2, \ldots, a_n$ in $F$.
2. The only solution to $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{0}$ is $c_1 = c_2 = \cdots = c_n = 0$.

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Remember our example of a set which was not linearly independent: $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. We have $\vec{u}_1 + \vec{u}_2 - \vec{u}_3 = \vec{0}$.

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**Proof:** $\texttt{NOT}(2) \implies \texttt{NOT}(1)$: If $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = \vec{0}$ then $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n = 0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n$, so we have two ways to write $\vec{0}$ as a linear combination of the $\vec{u}_i$.

$\texttt{NOT}(1) \implies \texttt{NOT}(2)$: Suppose that $a_1\vec{u}_1 + \cdots + a_n\vec{u}_n = b_1\vec{u}_1 + \cdots + b_n\vec{u}_n$ for $(a_1, a_2, \ldots, a_n) \neq (b_1, b_2, \ldots, b_n)$. Then

$$(a_1 - b_1)\vec{u}_1 + (a_2 - b_2)\vec{u}_2 + \cdots + (a_n - b_n)\vec{u}_n = \vec{0}. \qquad \square$$

## More on spanning

If $V$ is any vector space and $X$ is a subset of $V$, the ***span*** of $X$ is the set of all linear combinations $\sum a_i\vec{v}_i$ for scalars $a_1$, $a_2$, $\ldots$, $a_n$ and for $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$ in $X$.

We write $\mathrm{Span}(X)$ for the span of $X$. $\mathrm{Span}(X)$ is a subspace of $X$.

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**Lemma:** If $Y$ spans $V$, and $Y$ is in Span$(X)$, then $X$ spans $V$.

**Proof:** Let $\vec{v}$ be in $V$. Write $\vec{v} = \sum_i a_i \vec{y}_i$ for $\vec{y}_i$ in $Y$. Write $\vec{y}_i = \sum_j b_{ij} \vec{x}_j$. Then

$$\vec{v} = \sum_i a_i \Big( \sum_j b_{ij} \vec{x}_j \Big) = \sum_j \Big( \sum_i a_i b_{ij} \Big) \vec{x}_j .$$

So $\vec{v}$ is a linear combination of vectors $\vec{x}_j$ in $X$. $\square$

Since we want to work with infinite dimensional vector spaces, we also have definitions for infinite sets. Let $V$ be a vector space and let $X$ be a subset of $V$.

**Definition:** We say that $X$ is ***linearly independent*** if there is not a finite subset $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ of $X$ for which there are coefficients $c_1$, $c_2$, ..., $c_n$, not all zero, with $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$.

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Note that we never talk about infinite sums of vectors. If our vector space is $F^\infty$, we would **not** say

$$(1,1,1,\dots) = (1,0,0,\dots) + (0,1,0,\dots) + (0,0,1,\dots) + \cdots$$

because we wouldn't write down the right hand side at all.

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**Definition:** We say that $X$ is a ***basis*** of $V$ if it is both linearly independent and spans $V$.

To repeat again:

**A linearly independent set** can express each vector in **at most** one way. It is **not too big**.

**A spanning set** can express each vector in **at least** one way. It is **not too small**.

**A basis** can express each vector in **at most** one way. It is **just right**.

**Key Lemma:** Let $V$ be a vector space. Let $\vec{u}_1$, $\vec{u}_2$, $\ldots$, $\vec{u}_p$ be a spanning set, and let $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_q$ be linearly independent. Then $p \geq q$.

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Now, suppose for the sake of contradiction that $p < q$. Consider the $p$ equations in $q$ variables:

$$\begin{array}{ccccccccc} A_{11}c_1 & + & A_{12}c_2 & + & \cdots & + & A_{1q}c_q & = & 0 \\ A_{21}c_1 & + & A_{22}c_2 & + & \cdots & + & A_{2q}c_q & = & 0 \\ & & & & & & & \vdots & \\ A_{p1}c_1 & + & A_{p2}c_2 & + & \cdots & + & A_{pq}c_q & = & 0 \end{array}$$

Since $p < q$, they have a nonzero solution.

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Since $p < q$, they have a nonzero solution. Then,

$$\sum_j c_j\vec{v}_j = \sum_j c_j\Big(\sum_i A_{ij}\vec{u}_i\Big) = \sum_i \Big(\sum_j A_{ij}c_j\Big)\vec{u}_i = \sum_i 0\vec{u}_i = \vec{0}.$$

This contradicts that the $\vec{v}$'s are supposed to be linearly independent. $\square$

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**In other words:** Any spanning set is larger than any linearly independent set.

**Corollary:** If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q$ are both bases, then $p = q$.

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**Proof:** Since $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p$ is a basis, it is a spanning set; since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q$ is a basis, it is linearly independent. So $p \geq q$.

But, switching the roles of $\vec{u}$ and $\vec{v}$, we also have $p \leq q$. $\square$

**Corollary:** If $\vec{u}_1$, $\vec{u}_2$, ..., $\vec{u}_p$ and $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_q$ are both bases, then $p = q$.

We define the ***dimension*** of a vector space to be the number of elements in any basis. We say that a vector space is ***finite dimensional*** if it has a finite basis.

## Which vector spaces have bases?

To see that there is an issue, notice that $\mathbb{R}$ is a $\mathbb{Q}$ vector space. It is hard to imagine that there is some set of real number $B$ such that every real number is expressible in exactly one way as a rational linear combination of numbers from $B$.

To give another example, let $\mathbb{R}^\infty$ be the vector space of all sequences $(a_1, a_2, a_3, \ldots)$ of real numbers. The vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, 1, \ldots)$ are linearly independent, but they don't span. It is not at all clear whether we could expand this list to give a basis.

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Whether or not you think all vector spaces have bases comes down to a question about the foundations of mathematics; if you believe in a claim called the "Axiom of Choice", then they do, otherwise, it isn't clear.

## Which vector spaces have bases?

Without addressing this issue, here are two theorems which say that, in finite dimensions, there is no issue:

**Theorem** Let $V$ be a finite dimensional vector space and let $W$ be a subspace of $V$. Then $W$ has a finite basis.

**Theorem** Let $V$ be a vector space which has a finite spanning set. Then $V$ has a finite basis.

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Let $\vec{w}$ be in $W$. If $\vec{w}$ is in $B$, then clearly $\vec{w}$ is in $\mathrm{Span}(B)$.

If $\vec{w}$ is not in $B$, then $B \cup \{\vec{w}\}$ must be linearly dependent. Say

$$a\vec{w} + \sum c_i \vec{b}_i = \vec{0}$$

for $\vec{b}_i$ in $B$. Moreover, $a$ must be nonzero, as $B$ is linearly independent. Then

$$\vec{w} = -\tfrac{1}{a} \sum c_i \vec{b}_i$$

so $\vec{w}$ is in $\mathrm{Span}(B)$, as required. $\square$

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First, we show that $S$ is in $\mathrm{Span}(B)$. Let $\vec{s}$ be in $S$. As before, if $\vec{s}$ is in $B$, then obviously $\vec{s} \in \mathrm{Span}(B)$. And, if $\vec{s}$ is not in $B$, then $B \cup \{\vec{s}\}$ is linearly dependent and we use this to show that $\vec{s}$ is in $\mathrm{Span}(B)$.

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So $S$ is in Span$(B)$, and $\text{Span}(S) = V$. But then $B$ spans $V$, from our lemma from before. $\square$

**Remark:** These last two theorems were one of the first times we really needed to divide.

Suppose that we tried to do linear algebra with our scalars being the integers $\mathbb{Z}$. Let $V = \mathbb{Z}^2$, and consider $\vec{v}_1 = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\vec{v}_2 = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and $\vec{v}_3 = \left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$. Then one can check that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ span $\mathbb{Z}^2$. (Hint: $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = 3\vec{v}_1 - \vec{v}_2 - \vec{v}_3$, $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = -2\vec{v}_1 + \vec{v}_2 + \vec{v}_3$.) They are not linearly independent, since $6\vec{v}_1 = 3\vec{v}_2 + 2\vec{v}_3$.

However, no two of the $\vec{v}_i$ span $\mathbb{Z}^2$! If $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ is in $\mathrm{Span}(\vec{v}_1, \vec{v}_2)$ then $x - y$ is divisible by 2; if $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ is in $\mathrm{Span}(\vec{v}_1, \vec{v}_3)$ then $x - y$ is divisible by 3 and, if $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$ is in $\mathrm{Span}(\vec{v}_2, \vec{v}_3)$, then $x$ is even and $y$ is divisible by 3. So none of these pairs spans all of $\mathbb{Z}^2$.

This is a first example to show that linear algebra over a field is much nicer than linear algebra over what is called a ring. If you like studying hard, interesting, things, take commutative algebra! If you like studying easy, useful things, study linear algebra!