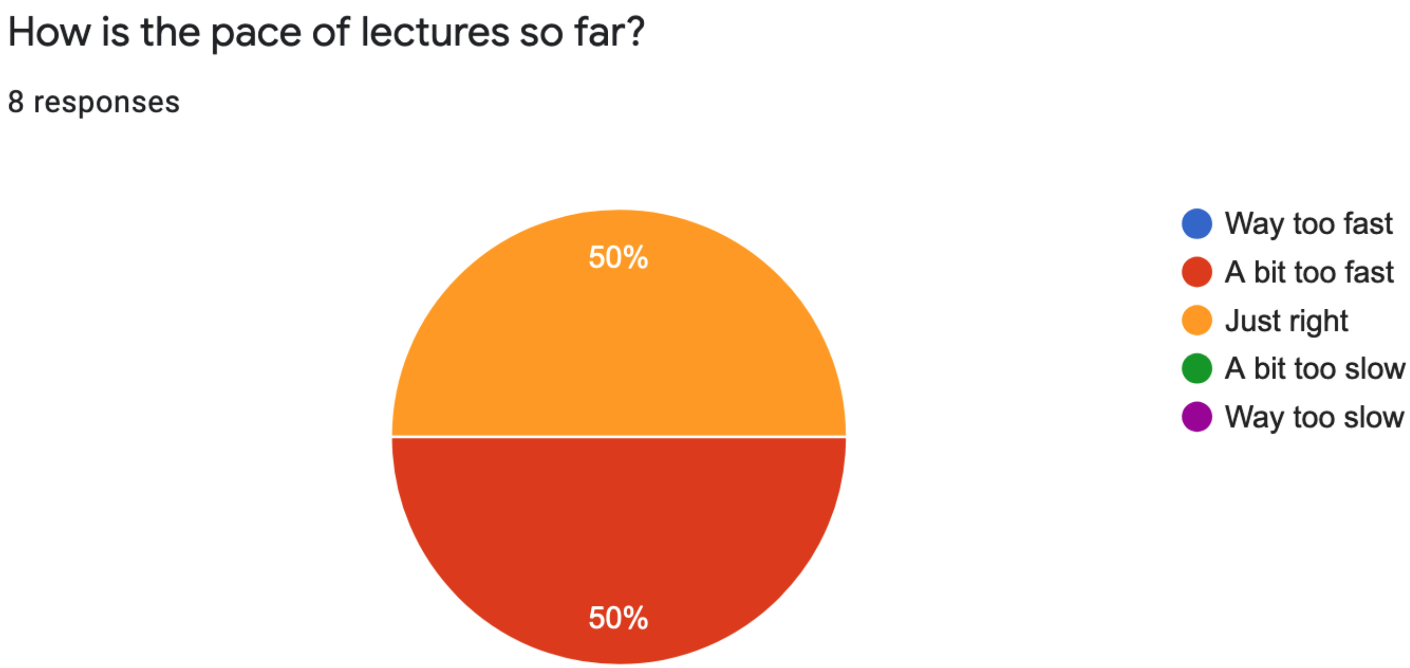
Bases of image and kernel

From the poll:

“I’m personally a big fan of when you handwrite things on the board, I feel like I can follow the pace a lot more naturally.”

“Hoping for more computational examples in lecture to help me to better understand how to apply what I have been learning.”



Also from the poll

"I am actually confused about the definition of field."

Bases of image and kernel

Two basis ways we will describe subspaces:

- As spans of some list of vectors. In other words, as an image.
- By defining equations. In other words, by a kernel.

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A finite dimensional example: Those vectors in $\mathbb{R}^3$ which are a linear combination of $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

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Note that this is also

$$\operatorname{Image} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

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An infinite dimensional example: The set of polynomials in $\mathbb{R}[x]$ which are of the form $(x^2-1)f(x)+(x^3-1)g(x)$.

Note that this is the image of the $\mathbb{R}$-linear map from $\mathbb{R}[x]^2 \to \mathbb{R}[x]$ sending $(f(x), g(x))$ to $(x^2-1)f(x)+(x^3-1)g(x)$.

By defining equations. In other words, by a kernel.

A finite dimensional example: Those vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ with $x + y + z = 0$.

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$$\operatorname{Ker}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

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An infinite dimensional example: Those polynomials $f(x)$ with $f(1) = 0$. Note that this is the kernel of the $\mathbb{R}$-linear map $\mathbb{R}[x] \to \mathbb{R}$ sending $f(x)$ to $f(1)$.

- As spans of some list of vectors. In other words, as an image.
- By defining equations. In other words, by a kernel.

How do we find a basis for each one? How do we switch from one to the other? (This is textbook 2.6, if you want to see another presentation.)

Suppose that we have a subspace L of F^m, described as $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$. How do we find a basis?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 2 \\ 10 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \ \vec{v}_4 = \begin{bmatrix} 11 \\ 21 \\ 11 \end{bmatrix}, \ \vec{v}_5 = \begin{bmatrix} 8 \\ 23 \\ 8 \end{bmatrix}$$

Suggestions? Observations?

One method: Column reduction

$$\begin{bmatrix} 1 & 2 & 4 & 11 & 8 \\ 5 & 10 & 3 & 21 & 23 \\ 1 & 2 & 4 & 11 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Claim: The nonzero columns of the column reduction are a basis.

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Spanning: We know that column operations don't change the image, and the zero columns don't contribute to the image, so the nonzero columns of the row reduction span.

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Linear independence: Each column has a pivot 1 in a row where all the other entries are zero.

Note that the dimension of Image(A) is the number of pivot 1's in the column reduction of A.

This method is very good for giving defining equations for Image(A). For each free row, we get a formula expressing it in terms of the pivot rows.

$$\mathrm{Image}\begin{bmatrix} 1 & 2 & 4 & 11 & 8 \\ 5 & 10 & 3 & 21 & 23 \\ 1 & 2 & 4 & 11 & 8 \end{bmatrix} = \mathrm{Image}\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

Can you see how to express x_3 in terms of x_1 and x_2 in our example? Can you write this space as a kernel?

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$$\left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] : x = z \right\} = \operatorname{Ker}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}.$$

So we now how to find a basis for a subspace given as an image, and how to write that space as a kernel.

Another way: row reduction

$$A = \begin{bmatrix} 1 & 2 & 4 & 11 & 8 \\ 5 & 10 & 3 & 21 & 23 \\ 1 & 2 & 4 & 11 & 8 \end{bmatrix} \quad \rightsquigarrow \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ 5 \\ 1 \end{smallmatrix}\right], \; \vec{v}_2 = \left[\begin{smallmatrix} 2 \\ 10 \\ 2 \end{smallmatrix}\right], \; \vec{v}_3 = \left[\begin{smallmatrix} 4 \\ 3 \\ 4 \end{smallmatrix}\right], \; \vec{v}_4 = \left[\begin{smallmatrix} 11 \\ 21 \\ 11 \end{smallmatrix}\right], \; \vec{v}_5 = \left[\begin{smallmatrix} 8 \\ 23 \\ 8 \end{smallmatrix}\right].$$

Problem Write $\vec{v}_2$, $\vec{v}_4$ and $\vec{v}_5$ as a linear combination of the vectors $\vec{v}_1$ and $\vec{v}_3$.

$$A = \begin{bmatrix} 1 & 2 & 4 & 11 & 8 \\ 5 & 10 & 3 & 21 & 23 \\ 1 & 2 & 4 & 11 & 8 \end{bmatrix} \quad \rightsquigarrow \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We saw before that A and R have the same kernel. Another way to say this is that the linear relations between the $\vec{a}_i$ are the same as the linear relations between the $\vec{r}_i$.

So a subset $\vec{a}_{i_1}$, $\vec{a}_{i_2}$, ..., $\vec{a}_{i_k}$ of the $\vec{a}_i$ is linearly independent if and only if the corresponding subset of the $\vec{r}_{i_1}$, $\vec{r}_{i_2}$, ..., $\vec{r}_{i_k}$ of the $\vec{r}_i$ is linearly independent.

And $\vec{a}_j$ is in $\mathrm{Span}(\vec{a}_{i_1}, \vec{a}_{i_2}, \ldots, \vec{a}_{i_k})$ if and only if $\vec{r}_j$ is in $\mathrm{Span}(\vec{r}_{i_1}, \vec{r}_{i_2}, \ldots, \vec{r}_{i_k})$.

So $\vec{a}_{i_1}$, $\vec{a}_{i_2}$, ..., $\vec{a}_{i_k}$ is a basis of $\mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ if and only if $\vec{r}_{i_1}$, $\vec{r}_{i_2}$, ..., $\vec{r}_{i_k}$ is a basis of $\mathrm{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)$!

$$A = \begin{bmatrix} 1 & 2 & 4 & 11 & 8 \\ 5 & 10 & 3 & 21 & 23 \\ 1 & 2 & 4 & 11 & 8 \end{bmatrix} \quad \rightsquigarrow \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{a}_{i_1}$, $\vec{a}_{i_2}$, ..., $\vec{a}_{i_k}$ is a basis of $\mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$ if and only if $\vec{r}_{i_1}$, $\vec{r}_{i_2}$, ..., $\vec{r}_{i_k}$ is a basis of $\mathrm{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)$.

But it is obvious that the pivot columns are a basis for $\mathrm{Span}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)$! So the $\vec{a}$'s which are in the pivot positions are a basis of $\mathrm{Span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$.

One nice thing about this method is that the basis that we get is a subset of $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$.

Note that the number of elements in our basis is the number of pivot columns of the row reduction.

So the number of pivots of the row reduction is the same as the number of pivots of the column reduction. In other words, the row space and the column space have the same dimension.

What if we are given a space by some linear equations? For example, what if we want a basis for the set of vectors (w, x, y, z) with

$$\begin{array}{ccccccccc} w & + & x & + & 2y & + & 3z & = & 0 \\ w & + & x & + & 3y & + & 5z & = & 0 \end{array}?$$

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This is the same as asking for the kernel of a matrix: In this case, $\operatorname{Ker}\left[\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 5 \end{smallmatrix}\right]$.

As we saw before, a matrix and its row reduction have the same kernel. So it is enough to understand the kernel of a row reduced matrix.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 1 & 0 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}.$$

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As we discussed before, we can compute use the row reduced form to compute the pivot variables in terms of the free variables:

$$x_1 = -x_2 + x_4 \qquad x_3 = -2x_4.$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

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This gives a basis of the kernel, since each element of the kernel can be written like this in a unique way. In particular, we have expressed this kernel as an image.