First topic: Coordinates

Let H be the plane $x + y + z = 0$ in $\mathbb{R}^3$.

• Give a basis $\vec{u}$, $\vec{v}$ of H. To test that your answer makes sense, write the vectors $\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$ as linear combinations of your basis elements.

• Let T be the linear map $(x, y, z) \longrightarrow (y, z, x)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$. For a general vector $p\vec{u} + q\vec{v}$, written in your basis, compute formulas for r and s such that

$$r\vec{u} + s\vec{v} = T\left(p\vec{u} + q\vec{v}\right).$$

When we introduced bases, we said that a basis allows us to treat a finite dimensional vector space V like F^n. How does this work?

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So there is a unique list of coefficients $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ for which $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$. We call $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ the "coefficients of $\vec{v}$ in the basis $\mathcal{B}$". We denote it as $[\vec{v}]_{\mathcal{B}}$.

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For example, let V be the plane $x + y + z = 0$ in $\mathbb{R}^3$ and let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Then $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ in the basis $\mathcal{B}$.

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Remark: Every textbook seems to have a different notation for this concept. This is our book's choice. If it were up to me, I'd go with ${}_{\mathcal{B}}[\vec{v}]$.

By the definition of a basis, we get a bijection $V \longrightarrow F^n$. We send $\vec{v}$ to $[\vec{v}]_{\mathcal{B}}$ and, in reverse, we send $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ to $\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$.

This bijection is a linear transformation between V and F^n.

Vocabulary: An invertible linear transformation is called an ***isomorphism***.

This bijection is a linear transformation between V and F^n. Let's check that this map is linear:

Proof: First, suppose that $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$ and $\vec{w} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$. So $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. Then we have

$$\vec{v} + \vec{w} = \left(c_1\vec{b}_1 + \cdots + c_n\vec{b}_n\right) + \left(d_1\vec{b}_1 + \cdots + d_n\vec{b}_n\right) = (c_1 + d_1)\vec{b}_1 + \cdots + (c_n + d_n)\vec{b}_n$$

so

$$[\vec{v} + \vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix}.$$

We also check scalar multiplication. So the map $V \to F^n$ is linear.

We could directly check that the inverse map is linear. But it is better to show:

Theorem: Let $T : U \to V$ be a bijective map of vector spaces. If T is a linear transformation, then so is T^{-1}.

Proof: Let $\vec{v}_1$ and $\vec{v}_2$ be in V, and let $\vec{u}_1 = T^{-1}(\vec{v}_1)$ and $\vec{u}_2 = T^{-1}(\vec{v}_2)$. We need to show that $T^{-1}(\vec{v}_1 + \vec{v}_2) = \vec{u}_1 + \vec{u}_2$.

Since T is a bijection, this is the same as showing that $\vec{v}_1 + \vec{v}_2 = T(\vec{u}_1 + \vec{u}_2)$. But this just follows by linearity of T:

$$T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2) = \vec{v}_1 + \vec{v}_2.$$

The argument for scalar multiplication is similar. $\square$

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In summary: A basis gives us an isomorphism between V and F^n, and coordinates are the explicit way we write this isomorphism.

Let's see how linear transformations work in coordinates (Textbook 3.4). Let $T : W \to V$ be a linear transformation. Let $\mathcal{W} = (\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n)$ be a basis of W and let $\mathcal{V} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ be a basis of V.

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Then

$$T(\vec{w}_j) = \sum B_{ij} \vec{v}_i$$

for some unique scalars B_{ij}. We call the matrix B "the linear transformation T in the coordinates of $\mathcal{V}$ and $\mathcal{W}$" and call it ${}_{\mathcal{V}}[T]_{\mathcal{W}}$.

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The matrix B determines the transformation. For any vector $\sum c_j \vec{w}_j$ in W, we have

$$T\left(\sum_j c_j \vec{w}_j\right) = \sum_j c_j T(\vec{w}_j) = \sum_j c_j \sum_i B_{ij}\vec{v}_i = \sum_i \left(\sum_j B_{ij}c_j\right)\vec{v}_i$$

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We can summarize this formula as

$$[T\vec{x}]_{\mathcal{V}} = \left({}_{\mathcal{V}}[T]_{\mathcal{W}}\right).$$

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We similarly have

$${}_{\mathcal{U}}[ST]_{\mathcal{W}} = \left({}_{\mathcal{U}}[S]_{\mathcal{V}}\right)\left({}_{\mathcal{V}}[T]_{\mathcal{W}}\right).$$

Let's go back to our example from the start of class:

Let H be the plane $x+y+z=0$ in $\mathbb{R}^3$. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and let $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Let T be the transformation $(x,y,z) \longrightarrow (y,z,x)$. So

$$\begin{aligned} T\vec{v}_1 &= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} &= \vec{v}_2 \\ T\vec{v}_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} &= -\vec{v}_1 - \vec{v}_2 \end{aligned}$$

So the transformation T, in the basis $(\vec{v}_1, \vec{v}_2)$, is

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

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Consider the following problem: Compute

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{100} \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

(Why? Wait until we get to eigenvectors for real.)

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Attacked directly, this is a mess. But let's work in the basis $\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\vec{v}_2 = \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]$. What is the matrix of $\left[\begin{smallmatrix} 1 & 1 \\ 4 & 1 \end{smallmatrix}\right]$ in this basis?

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We have $\left[\begin{smallmatrix} 1 & 1 \\ 4 & 1 \end{smallmatrix}\right] \vec{v}_1 = \left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right] = 3\vec{v}_1$ and $\left[\begin{smallmatrix} 1 & 1 \\ 4 & 1 \end{smallmatrix}\right] \vec{v}_2 = \left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right] = -\vec{v}_2$. So, in this basis, the matrix is $\left[\begin{smallmatrix} 3 & 0 \\ 0 & -1 \end{smallmatrix}\right]$.

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And what are the coordinates of $\left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right]$ in this basis? $\left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right] = 2\vec{v}_1 + \vec{v}_2$, so $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$.

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So, in these coordinates, we are trying to compute

$$\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}^{100}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}3^{100}\\ 2\end{bmatrix}.$$

So the answer to the original question is that

$$\begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}^{100}\begin{bmatrix}1\\ 6\end{bmatrix} = 3^{100}\vec{v}_1 + \vec{v}_2 = 3^{100}\begin{bmatrix}1\\ 2\end{bmatrix} + \begin{bmatrix}-1\\ 2\end{bmatrix}.$$

Second topic: Direct sums and quotient spaces

We start with a homework problem: Let V be a vector space and let X and Y be subspaces. Show that the following are equivalent:

1. Every vector in V can be written in exactly one way as $\vec{x} + \vec{y}$ for $\vec{x} \in X$ and $\vec{y} \in Y$.
2. Every vector in V can be written as $\vec{x} + \vec{y}$ for $\vec{x} \in X$ and $\vec{y} \in Y$, and $X \cap Y = \{0\}$.

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Proof: In either case, we are assuming that every vector can be written as $\vec{x} + \vec{y}$.

(1) $\implies$ (2): If $\vec{u}$ is in $X \cap Y$, then $\vec{x} + \vec{y} = (\vec{x} + \vec{u}) + (\vec{y} - \vec{u})$. This would give multiple formulas for the same vector unless $\vec{u} = \vec{0}$.

(2) $\implies$ (1): Suppose, to the contrary, that $\vec{x}_1 + \vec{y}_1 = \vec{x}_2 + \vec{y}_2$. Then $\vec{x}_1 - \vec{x}_2 = \vec{y}_1 - \vec{y}_2$, so assumption (2) tells us that $\vec{x}_1 - \vec{x}_2 = \vec{y}_1 - \vec{y}_2 = 0$, and we have $\vec{x}_1 = \vec{x}_2$ and $\vec{y}_1 = \vec{y}_2$. $\square$

In this case, we'll say that $V = X \oplus Y$.

For example, $\mathbb{R}^3 = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \} \oplus \{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \}$.

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If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ is a basis of X, and $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$ is a basis of Y, then $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$ is a basis of V. In particular, $\dim V = \dim X + \dim Y$.

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If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ is a basis of X, and $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$ is a basis of Y, then $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n$ is a basis of V. In particular, $\dim V = \dim X + \dim Y$.

So, when we write vectors in the coordinates of this basis, the X-entries come first and then the Y-entries. Similarly, if $V_1 = X_1 \oplus Y_1$ and $V_2 = X_2 \oplus Y_2$, then linear transformations $V_1 \to V_2$ are given by block matrices.

$$\left[\begin{array}{c|c} X_1 \to X_2 & Y_1 \to X_2 \\ \hline X_1 \to Y_2 & Y_1 \to Y_2 \end{array}\right].$$

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Let X and Y be two vector spaces over the same field F. We define the vector space $X \boxplus Y$ as follows:

- The elements of $X \boxplus Y$ are ordered pairs $(\vec{x}, \vec{y})$ with $\vec{x} \in X$ and $\vec{y} \in Y$.
- Addition is defined as $(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2) = (\vec{x}_1 + \vec{x}_2, \vec{y}_1 + \vec{y}_2)$.
- Scalar multiplication is defined as $c(\vec{x}_1, \vec{y}_1) = (c\vec{x}_1, c\vec{y}_1)$.

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So, if X and Y are both subspaces of V and $V = X \oplus Y$, then $X \boxplus Y$ is isomorphic to $X \oplus Y$, by $(\vec{x}, \vec{y}) \mapsto \vec{x} + \vec{y}$. But we are allowed to talk about $X \boxplus Y$ without starting with a subspace that X and Y are both contained in.

Finally, quotient spaces. Let V be a vector space and let X be a subspace. Then there are many different subspaces Y with $V = X \oplus Y$.

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One answer, for finite dimensional real vector spaces, is $X^{\perp}$, but that isn't ours.

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Define $\vec{v}_1 \equiv \vec{v}_2 \mod X$ if $\vec{v}_1 - \vec{v}_2 \in X$. Check that

- If $\vec{v}_1 \equiv \vec{v}_2$ and $\vec{w}_1 \equiv \vec{w}_2$ then $\vec{v}_1 + \vec{w}_1 \equiv \vec{v}_2 + \vec{w}_2$.
- If $\vec{v}_1 \equiv \vec{v}_2$ and c is a scalar then $c\vec{v}_1 \equiv c\vec{v}_2$.

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The elements of V/X are the equivalence classes for V/X, with addition and scalar multiplication defined as above.

If $V = X \oplus Y$, then $Y \longrightarrow V \longrightarrow V/X$ is an isomorphism.