Let V be a vector space and let $A : V \to V$ be a linear transformation. Let $\vec{v} \in V$ be a vector and λ a scalar such that $A\vec{v} = \lambda\vec{v}$. We say that $\vec{v}$ is a λ-***eigenvector*** of A. For a scalar λ, we will call the set of all λ-eigenvectors of A the λ-eigenspace. In other words, the λ-eigenspace is $\{\vec{v} \in V : A\vec{v} = \lambda\vec{v}\}$.

Wake up: Check that the λ-eigenspace is a subspace of V.

If the λ-eigenspace is not $\{\vec{0}\}$, we call λ an ***eigenvalue*** of A. Your textbook uses the old fashioned words "characteristic values" and "characteristic vector".

Problem 1.
(1) Show that λ is an eigenvalue of A if and only if $\mathrm{Ker}(A - \lambda \mathrm{Id}_n) \neq \{\vec{0}\}$.
(2) Show that λ is an eigenvalue of A if and only if $\det(A - \lambda \mathrm{Id}_n) = 0$.

The polynomial $\det(A - t\mathrm{Id})$ is called the ***characteristic polynomial*** of A. So the roots of the characteristic polynomial (in F) are the eigenvalues.

Problem 2. Let $A = \left[\begin{smallmatrix} 7 & 4 \\ -2 & 1 \end{smallmatrix}\right]$.
(1) What are the eigenvalues of A?
(2) What are the eigenvectors of A?

Problem 3.
(1) Let $\lambda_1 \neq \lambda_2$. Let $\vec{v}_1$ be a nonzero λ_1-eigenvector and let $\vec{v}_2$ be a nonzero λ_2-eigenvector. Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. (This is straightforward.)
(2) Let λ_1, λ_2 and λ_3 be distinct scalars.. Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ be nonzero eigenvectors for λ_1, λ_2 and λ_3. Show that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent. (This takes a bit more thought.)
(3) In general, let λ_1, λ_2, $\ldots$, λ_k be scalars and let $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_k$ be corrresponding nonzero eigenvectors. Show that $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_k$ are linearly independent.

To repeat, let V be a finite dimensional vector space and let $A : V \to V$ be a linear transformation. The ***characteristic polynomial*** of A is $\det(x\mathrm{Id} - A)$. We'll denote it by $\chi_A(x)$.

Problem 4. Let A be an $n \times n$ square matrix of the form $\begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$ where P is $k \times k$ and R is $(n-k) \times (n-k)$.

(1) Show that $\det(A) = \det(P)\det(R)$.
(2) Show that $\chi_A(x) = \chi_P(x)\chi_R(x)$

Problem 5. To repeat, let V be a finite dimensional vector space and let $A : V \to V$ be a linear transformation. Let U be a subspace of V such that A maps U to U. Show that $\chi_{A|_U}(x)$ divides $\chi_A(x)$.

Problem 6. (1) Show that the characteristic polynomial of $\begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix}$ is $x^2 + ax + b$.

(2) Show that the characteristic polynomial of $\begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix}$ is $x^3 + ax^2 + bx + c$.
(3) Show that the characteristic polynomial of

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ & & & & \vdots \\ 0 & 0 & \ddots & 0 & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

is $x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$.