The division algorithm for polynomials: If a(x) and b(x) are polynomials, with b not equal to 0, then there exist unique polynomials q(x) and r(x) such that

a(x) = b(x) q(x) + r(x)

with deg r(x) < deg b(x).

quotient

remainder

Theorem: Any polynomal can be written uniquely (up to scalar multiple) as a product of irreducible polynomials.

Proof: Clearly, any polynomial can be written as a product of irreducibles: If it isn't irreducible already, split it into two smaller factors. Inductively, those smaller factors are products of irreducibles.

We'll prove uniqueness by induction on deg(f).
Linear polynomials are already irreducible.

Suppose that the factorization of f is not unique.
Suppose that f(x) = a_1(x) a_2(x) … a_m(x) = b_1(x) b_2(x) … b_n(x).

Let b_n(x) have the smallest degree of the a_i's and b_j's.
We can write a_i(x) = q_i(x) b_n(x) + r_i(x) for some q_i and r_i.

Case 1: One of the r_i's is 0.
In this case, b_n(x) | a_i(x). Since a_i(x) is irreducible, this means that b_n(x) = a_i(x) up to a scalar.
Cancel a_i(x) from the LHS and b_n(x) from the RHS and get
a1(x) a2(x) … a_{i-1}(x) a_{i+1}(x) … a_m(x) =
b1(x) b2(x) … b_{n-1}(x).
Inductively, these factorizations are equal, so the original factorizations matched.

Case 2: None of the r_i(x) are 0.

Since deg r_i < deg b_n <= deg a_i, we know that the r_i's are smaller than the a_i.

We have:
(q1*bn+r1)(q2*bn+r2)….(qm*bn+rm) = b1 b2 … bn.

bn*(stuff) + r1*r2*…*rm = (b1 b2 … b_{n-1}) b_n

So b_n divides r1 r2 … rm.
Now, b_n is irreducible, and deg r_i < deg a_i, so by induction, we have unique factorization of the product r1 r2 … rm.
In particular, bn must divide some ri.
But deg r_i < deg b_n, so this means r_i = 0, we are back in Case 1.

Consequence: Let f(x) and g(x) be polynomials of degrees a and b with no common factor. Then there does not exist any solution, other than u(x) = v(x)=0, to the equations

f(x) u(x) = g(x) v(x)

with deg u < b and deg v < a .

If we take u = g and v = f, then we have

f(x) u(x) = g(x) v(x). But
then deg u(x) = b and deg v(x) = a.

If there were a common factor, h(x), then we could take u(x) = f(x)/h(x) and v(x) = g(x)/h(x).

Proof: If f(x) u(x) = g(x) v(x) then every irrducible factor of g(x) must also appear in u(x). So g(x) | u(x), and deg u(x) >= deg g(x) = b.

Similarly, every irred factor of f(x) must appear in v(x), so f(x) | v(x) and deg v(x) >= deg f = a.

In other words, if

(x^2+1) u(x) = (x^3-2) v(x), then

we must have deg(v) >= 2 and deg u >= 3.