Here is our goal:

Final version: Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. Let $g(x)$ be a nonzero polynomial with $g(T) = 0$, and let $g(x)$ factor as $f_1(x)f_2(x)\cdots f_r(x)$ with $\mathrm{GCD}(f_i, f_j) = 1$. Let $W_i = \mathrm{Ker}(f_i(T))$.

Then $V = \bigoplus W_i$ and the transformation T takes W_i to W_i.

Moreover, if $g(x) = \chi_T(x)$, then $\chi_{T|_{W_i}}(x) = f_i(x)$. If $g(x) = m_T(x)$, then $m_{T|_{W_i}}(x) = f_i(x)$.

Problem 1. Recall that we defined $W_i = \mathrm{Ker}(f_i(T))$. Show that T takes W_i into W_i.

We'll now do the case of two polynomials: $g(x) = f_1(x)f_2(x)$. So we want to show that $V = W_1 \oplus W_2$.

Problem 2. Notice that we have $g(A) = f_1(A)f_2(A) = 0$. Show that $\dim \mathrm{Ker}(f_1(A)) + \dim \mathrm{Ker}(f_2(A)) \geq \dim V$.

Problem 3. Show that T takes $W_1 \cap W_2$ into $W_1 \cap W_2$.

Problem 4. Let $m(x)$ be the minimal polynomial of T restricted to $W_1 \cap W_2$.

(1) Show that $m(x)$ divides $\mathrm{GCD}(f_1(x), f_2(x))$.
(2) Now use that $\mathrm{GCD}(f_1(x), f_2(x)) = 1$ and deduce that $W_1 \cap W_2 = \{0\}$.
(3) For future use, deduce that $f_1(T)$, restricted to W_2, is invertible.

Problem 5. Put all the above parts together to deduce that $V = W_1 \oplus W_2$.

We now generalize to $g(x) = f_1(x)f_2(x)\cdots f_r(x)$.

Problem 6. Put $n_i = \dim \mathrm{Ker}(f_i(T)) = \dim W_i$. Show that $n_1 + n_2 + \cdots + n_r \geq \dim W_i$.

Problem 7. We will show by induction on k that $W_1 + W_2 + \cdots + W_k = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. So suppose that we already know that $W_1 + W_2 + \cdots + W_{k-1} = W_1 \oplus W_2 \oplus \cdots \oplus W_{k-1}$. We need to show that $(W_1 + W_2 + \cdots + W_{k-1}) \cap W_k = \{0\}$.

Let $m(x)$ be the minimal polynomial of T restricted to $(W_1 + W_2 + \cdots + W_{k-1}) \cap W_k$.

(1) Show that $m(x)$ divides $\mathrm{GCD}(f_1(x)f_2(x)\cdots f_{k-1}(x), f_k(x))$.
(2) Conclude that $(W_1 + W_2 + \cdots + W_{k-1}) \cap W_k = \{0\}$.

Problem 8. Put all the above parts together to deduce that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$.

Finally, we check off the claims about minimal polynomials.

Problem 9. Suppose that $m_T(x) = f_1(x)f_2(x)\cdots f_k(x)$ as above. Let $m_i(x)$ be the minimal polynomial of T restricted to W_i.

(1) Show that $m_i(x)$ divides $f_i(x)$.
(2) Show that $m(x)$ divides $m_1(x)m_2(x)\cdots m_k(x)$.
(3) Deduce that $m_i(x) = f_i(x)$.

The characteristic polynomial statement is a bit harder.