Let V be a real vector space with an inner product $B(\ ,\ )$. A list of vector $\vec{u}_1$, $\vec{u}_2$, … in V is called ***orthonormal*** if

$$B(\vec{u}_i, \vec{u}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Problem 1. Show that, if $\vec{u}_1$, $\vec{u}_2$, …, are orthonormal, then they are linearly independent.

Let $\vec{u}_1$, $\vec{u}_2$, …, $\vec{u}_n$ be an orthonormal basis of $X \subset V$. Define a linear map $p_X : V \to X$ by the formula:

$$p_X(\vec{v}) = \sum_{i=1}^{n} B(\vec{u}_i, \vec{v})\vec{u}_i.$$

Problem 2. Show that, for $\vec{x} \in X$, we have $p_X(\vec{x}) = \vec{x}$.

Problem 3. Show that, for $\vec{y} \in X^\perp$, we have $p_X(\vec{y}) = \vec{0}$.

Problem 4. Show that $V = X \oplus X^\perp$.

We'll also want to know the variant formula for p_X when the $\vec{u}_1$, $\vec{u}_2$, …, $\vec{u}_n$ are orthogonal, not orthonormal:

$$p_X(\vec{v}) = \sum_{i=1}^{n} \frac{B(\vec{u}_i, \vec{v})}{B(\vec{u}_i, \vec{u}_i)}\vec{u}_i.$$

Problem 5. Check that this is correct.