Let V be an n-dimensional vector space over a field F and let $A : V \to V$ be a linear transformation. Then we have defined a scalar $\det(A)$.
We can define it abstractly by

$$\omega(A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n) = (\det A)\omega(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$$

where ω is any non-zero alternating multilinear n-form.
Concretely, in any basis, we have

$$\det(A) = \sum_{\sigma \in S_n} \texttt{sign}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}.$$

We have the basic properties:

$$\det(AB) = \det(A)\det(B).$$

- Adding a scalar multiple of one row/column to another keeps the determinant the same.
- Multiplying a row/column by c rescales the determinant by c.
- Switching two rows/columns switches the sign of the determinant.
- We have $\det(A) = 0$ if and only if A is not invertible.

Wake up: What is

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}?$$

Problem 1. (1) Show that

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = A_{11} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13} \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}.$$

(2) Show that

$$A_{21} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{22} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{23} \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = 0.$$

Let $\widehat{A}_{ij}$ be the matrix A with the i-th row and the j-th column removed. We generalize the above computations to matrices of arbitrary size:

Problem 2. (1) For any row r, show that

$$\det(A) = (-1)^{r+1} \left(A_{r1} \det \widehat{A}_{r1} - A_{r2} \det \widehat{A}_{r2} + A_{r3} \det \widehat{A}_{r3} - \cdots + A_{rn} \det \widehat{A}_{rn} \right).$$

(2) If $q \neq r$, show that

$$A_{q1} \det \widehat{A}_{r1} - A_{q2} \det \widehat{A}_{r2} + A_{q3} \det \widehat{A}_{r3} - \cdots + A_{qn} \det \widehat{A}_{rn} = 0.$$

The adjoint matrix of A is defined by

$$\operatorname{adj}(J)_{ij} = (-1)^{i+j} \det \widehat{A}_{ji}.$$

(You saw this in homework problem 5.2.3.) For example,

$$\operatorname{adj} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & -(a_{21}a_{33}-a_{23}a_{31}) & a_{21}a_{32}-a_{22}a_{31} \\ -(a_{12}a_{33}-a_{13}a_{32}) & a_{11}a_{33}-a_{13}a_{31} & -(a_{11}a_{32}-a_{12}a_{31}) \\ a_{12}a_{23}-a_{13}a_{22} & -(a_{11}a_{23}-a_{13}a_{21}) & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}$$

Problem 3. Show that

$$\operatorname{adj}(A)A = \det(A)\mathrm{Id}_n.$$

Problem 4. If $\det(A) \neq 0$, show that $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$.

Problem 5. Let $\det A \neq 0$. We consider solving the equation

$$A\vec{x} = \vec{b}.$$

(1) Show that

$$x_i = \frac{1}{\det A} \sum_j \det(-1)^{i+j} \widehat{A}_{ji} x_j.$$

(2) (**Cramer's rule**) Let $A_i(b)$ be the matrix where we replace the i-th column of A with b. Show that

$$x_i = \frac{\det A_i(b)}{\det A}.$$