V a vector space over a field F

$$m(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) \quad \text{output in } F$$

A multilinear form is a function whose inputs are k vectors from V and whose outputis a scalar F which is linear in each variable.

$$m(\vec{x}+\vec{y}, \vec{v}_2, \ldots, \vec{v}_k) = m(\vec{x}, \vec{v}_2, \ldots, \vec{v}_k) + m(\vec{y}, \vec{v}_2, \ldots, \vec{v}_k)$$

$$m(c\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = c\, m(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$$

These conditions hold for any input position, not just the first.

If e1, e2, …, en is a basis of V, then m is uniquely determined by the n^k values

m(e_{i_1}, e_{i_2}, …, e_{i_k})

So the vector space of k-linear forms on V has dimension n^k.

Suppose we have an a-linear form alpha on V and a b-linear form beta. We can make an (a+b)-linear form by tensoring them together.

$$(\alpha \otimes \beta)(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{a+b}) =$$

$$\alpha(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_a)\beta(\vec{v}_{a+1}, \vec{v}_{a+2}, \ldots, \vec{v}_{a+b})$$

Let V have basis e1, e2.
Let a=b=1, so alpha and beta are in the dual space V*.
Let alpha = e1*+2 e2*
beta = 3 e1*+4 e2*.

$$(\alpha \otimes \beta)\left(\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix}\right) =$$

$$\alpha\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) \beta\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) = (p+2q)(3r+4s)$$

$$\boxed{\alpha\left(\begin{bmatrix} p \\ q \end{bmatrix}\right) = p\,\alpha\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + q\,\alpha\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = p \cdot 1 + q \cdot 2}$$

$$\alpha, \beta : V \longrightarrow F$$

$$\alpha \otimes \beta \longleftarrow$$ Bilinear form on V.

An alternating form is uniquely determined by its value on (e_{i_1}, e_{i_2}, …, e_{i_k}) for 1 <= i_1 < i_2 < … < i_k <= n .

So the vector space of alternating forms has dimension (n choose k)

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \longleftarrow$$ This is alternating (also known as skew-symmetric or anti-symmetric)

In particular, if k=n, the space of alternating forms is one dimensional, and gives the determinant.

In other words:

$$(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \beta(u)\alpha(v)$$

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \alpha_{\sigma(1)} \otimes \alpha_{\sigma(2)} \otimes \ldots \otimes \alpha_{\sigma(k)}.$$

sign(sigma) = (-1)^{# of pairs (i,j) with i<j and sigma(i)>sigma(j) }.

$$\alpha \wedge \beta \wedge \gamma = \alpha \otimes \beta \otimes \gamma - \alpha \otimes \gamma \otimes \beta - \beta \otimes \alpha \otimes \gamma + \beta \otimes \gamma \otimes \alpha + \gamma \otimes \alpha \otimes \beta - \gamma \otimes \beta \otimes \alpha$$

Let V be an n-dimensional vector space. Then there is a 1-dimensional space of alternating n-linear forms on V.

If we choose a basis e1, e2, …, en for V, then such an alternating form takes (v1, v2, …, vn) to

$$\det\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}.$$

All alternating forms are a multiple of this one.

Given T : V —> V, we get a scalar called det(V). If omega is any nonzero alternating n-linear form, then

omega(T v1, T v2, …, T vn) = det(T) omega(v1, v2, …, vn).

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Basic properties of det:

det(AB) = det(A) det(B).

Elementary row/column operations.

* Adding a multiple of one row/column to another doesn't change determinant.

* Rescaling a single row/column by a scalar c multiplies determinant by c.

* Switching two rows/columns changes the sign of the determinant.

For T : V —> V, we have det(T)=0 if and only T is not invertible. Equivalently, T has a kernel. Equivalently, T does not have image = V.

det(v1, v2, …, vn) = 0
if and only if v1, v2, …, vn are NOT a basis. Equivalently, v1,v2, …, vn are linearly dependent. Equivalently, they don't span V.

Let $e_1, e_2, \dots, e_n$ be the standard basis of V.

Then

$(e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)(v_1, v_2, \dots, v_n)$ is $\det(v_1, v_2, \dots, v_n)$.

Concretely, expanding this gives the formula for determinant as a sum of permutations.

Abstractly, we know that there is only a one dimensional space of alternating forms on V,
so $(e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)$
must be proportional to determinant, and it isn't hard to check what the scalar is.

$(e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)(v_1, v_2, \dots, v_n) =$
$((e_1^*) \wedge (e_2^* \wedge \dots \wedge e_n^*))(v_1, v_2, \dots, v_n) =$
$e_1^*(v_1)\ (e_2^* \wedge \dots \wedge e_n^*)(v_2, v_3, \dots, v_n)\ -$
$e_1^*(v_2)\ (e_2^* \wedge \dots \wedge e_n^*)(v_1, v_3, \dots, v_n)\ +$
$e_1^*(v_3)\ (e_2^* \wedge \dots \wedge e_n^*)(v_1, v_2, \dots, v_n)\ -$
$\dots +$
$e_1^*(v_n)\ (e_2^* \wedge \dots \wedge e_n^*)(v_1, v_2, \dots, v_{n-1})$

This is the row expansion of $\det(v_1, v_2, \dots, v_n)$ along row 1.

Cayley - Hamilton:

Let T : V ---> V be a linear transformation and let chi_T be its characteristic polynomial.

$$\chi_T(x) = \det(x\mathrm{Id} - T).$$

Then chi_T(T) = 0.

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\chi_T(x) = \det \begin{bmatrix} x-1 & -2 \\ -3 & x-4 \end{bmatrix}$$

$$= (x-1)(x-4) - (-2)(-3)$$

$$= x^2 - 5x + 4 - 6$$

$$= x^2 - 5x - 2$$

$$T^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$5T = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} \qquad 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T^2 - 5T + 2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \chi_A(x) = \det \begin{bmatrix} x-1 & -1 \\ -1 & x \end{bmatrix} =$$

$$x(x-1) - 1 = x^2 - x - 1$$

So A^2 - A - Id = 0.

So A^{n+2} - A^{n+1} - A^n = 0
or

A^{n+2} = A^{n+1} + A^n

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = A^2 + A$$

$$A^4 = A^3 + A^2 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

The powers of this matrix are matrices of Fibonacci numbers!

$$\begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix}$$

Special cases of Cayley-Hamilton:

If A is diagonalizable, may as well assume diagonal:

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\chi_A(x) = \det(x\,\mathrm{Id} - A) = \det \begin{bmatrix} x-\lambda_1 & & \\ & \ddots & \\ & & x-\lambda_n \end{bmatrix} = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

$$\chi_A(A) = \begin{bmatrix} (\lambda_1-\lambda_1)(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_n) & & & \\ & (\lambda_2-\lambda_1)(\lambda_2-\lambda_2)\cdots(\lambda_2-\lambda_n) & & \\ & & \ddots & \\ & & & (\lambda_n-\lambda_1)(\lambda_n-\lambda_2)\cdots(\lambda_n-\lambda_n) \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

Special case of Cayley-Hamilton:
Suppose there is a vector v such that $v, Tv, T^2 v, \ldots, T^{(n-1)} v$ form a basis of V.

In that case,

$$T^n(v) = c_{n-1} T^{n-1} v + \ldots + c_1 T v + c_0 v$$

for some scalars $c_0, c_1, \ldots, c_{n-1}$.

In this basis, the matrix of T is

$$\begin{bmatrix} 0 & 0 & & 0 & c_0 \\ 1 & 0 & & 0 & c_1 \\ 0 & 1 & & 0 & c_2 \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & & & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & & 1 & c_{n-1} \end{bmatrix}$$

The characteristic polynomial is

$$x^n - c_{n-1} x^{n-1} - \ldots - c_1 x - c_0 .$$

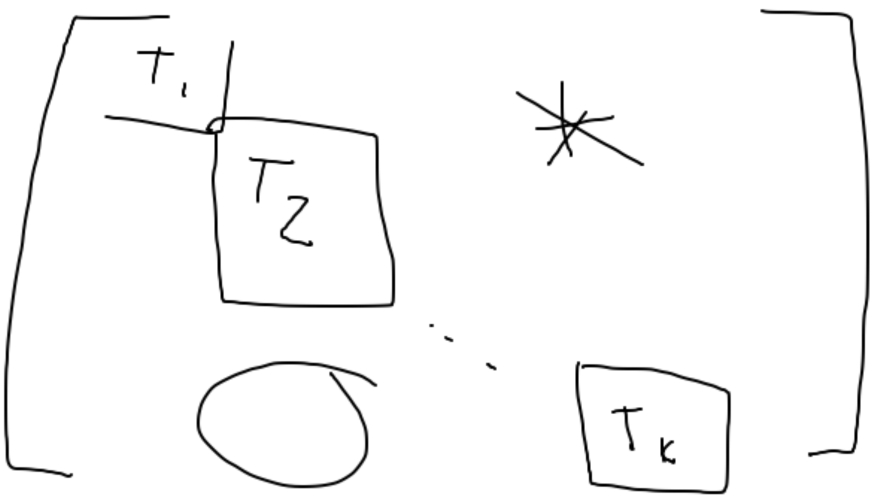
Want to check that

$$T^n - c_{n-1} T^{n-1} - \ldots - c_1 T - c_0 \cdot Id = 0.$$

Just need to check that it sends each basis vector to 0.

$T^n (T^j v) - c_{n-1} T^{n-1} (T^j v) - \ldots - c_1 T (T^j v) - c_0 \cdot (T^j v) =$
$T^{(n+j)} v - c_{n-1} T^{n-1+j} v - \ldots - c_0 T^j v =$
$(T^{(n+j)} - c_{n-1} T^{n-1+j} - \ldots - c_0 T^j) v =$
$T^j (T^n - c_{n-1} T^{n-1} - \ldots - c_0 \cdot Id) v =$
$T^j \cdot 0 = 0.$

General case, is that we show
that any linear transformation can
be put in the form



where each T_k has a
cyclic vector.

Just like in the diagonalizable case,

chi_T(x) = product of chi_{T_k}(x).

By the cyclic vector case,
chi_{T_k}(T_k) = 0.

We use this to show that
chi_T(T)=0.

Block upper triangularization —>
factorization of
characteristic polynomial

Primary decomposition theorem says that
(factorization of chi_T or minimal polynomial) —>
(block diagonalization)

Primary decomposition theorem.

Let V be finite dimensional vector space. Let T : V —> V be a linear transformation. Let g(x) be a polynomial with g(T) = 0.

Suppose g(x) factors as
f1(x) f2(x) … fk(x)
with GCD(fi(x), fj(x)) = 1 for i and j relatively prime.

Put Wi = Ker(fi). Then we have

* V is the direct sum of the W_i
* T maps W_i to W_i
* T restricted to W_i obeys f_i.

In coordinates, we can block diagonalize T. Each block gives T restricted to W_i.

Quotient spaces:

Let V be a vector space, W a subspace, V/W is the quotient space.

If v1, v2, …, vn is a basis for V such that
v1, v2, .., vk is a basis for W,
then v_{k+1}, v_{k+2}, …, v_{n} is a basis for V/W.

In particular, dim(V/W) = dim(V) - dim(W).

We always have a surjective linear map V ——> V/W whose kernel is W.

$$\begin{tikzcd} V \arrow[r] \arrow[d, "A"'] & V/W \arrow[dl, dashed] \\ X & \end{tikzcd}$$

If we have another linear map A: V —> X with A(W) = 0, then it factors through V/W

In particular, suppose that we have B: V —> V with B sending W to W. Then, in bases v1, v2, …, vn as before, B looks like

$$B = \begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$$

where P is the matrix of W—>W and R is the matrix of V/W —> V/W.