Vectors are n-tuples of numbers.

We add vectors (of the same dimension) coordinate by coordinate:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

We can also multiply a vector by a scalar:

$$c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

Matrices are rectangular arrays of numbers. An $m \times n$ matrix has m rows and n columns. Here is a 3×4 matrix:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}.$$

We add matrices (of the same dimensions) component by component:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \end{bmatrix}.$$

A matrix takes a vector as input and makes a new vector:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \end{bmatrix}.$$

This is a special case of the more general operation of matrix multiplication. We can multiply matrices when the number of columns of the first matrix equals the number of rows of the second:

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \cdot \begin{bmatrix} s & t & u & v \\ w & x & y & z \end{bmatrix} = \begin{bmatrix} as+bw & at+bx & au+by & av+bz \\ cs+dw & ct+dx & cu+dy & cv+dz \\ es+fw & et+fx & eu+fy & ev+fz \end{bmatrix}.$$

We write $0_{m\times n}$ for the $m \times n$ matrix which is all 0's (and drop the subscripts when they are clear from context) and write Id_n for the $n \times n$ matrix which is 1 on the diagonal and 0's everywhere else. Here are the key properties of matrix multiplication and addition:

$$\begin{array}{cc} A + 0 = 0 + A = A & \mathrm{Id}\ X = X \quad X\ \mathrm{Id} = X \\ & A0 = 0 \quad 0A = 0 \\ A + B = B + A & \\ (A+B)+C = A+(B+C) & (AB)C = A(BC) \\ A(B+C) = AB + AC & (X+Y)Z = XZ + YZ \end{array}$$

Note that AB is usually **not** equal to BA. In sophisticated language, matrices form a "noncommutative ring".