Question 1 (15 points)

We give a matrix M and its row reduction:

$$M = \begin{bmatrix} -1 & -2 & 2 & 5 & -1 \\ 2 & 4 & 0 & 2 & -1 \\ -1 & -2 & 2 & 5 & -2 \end{bmatrix} \qquad \text{rref}(M) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (2 points) What is the rank of M?

There are two pivot columns, so the rank is 2.

(b) (2 points) What is the dimension of the image of M? Explain how you know.

The dimension of the image is the same as the rank, so $\dim \text{Image}(M) = 2$.

(c) (3 points) What is the dimension of the kernel of M? Explain how you know.

By the rank-nullity theorem, the dimension of the kernel is the number of columns minus the rank, so $\dim \text{Ker}(M) = 5 - 2 = 3$.

(d) (4 points) Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the image of M? Explain why or why not.

No, it is not. One can see this systematically by row-reducing the matrix formed by adding the column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to M. A quicker route is to notice that every vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in Image(M) has $x_1 = x_3$.

(e) (4 points) Is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ in the kernel of M? Explain why or why not.

No, it is not. It is easier to check this using the row reduction of M, which has the same kernel. We compute

$$\text{rref}(M) \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 0 \end{bmatrix} \neq \vec{0}.$$

Question 2 (19 points)

A quadratic polynomial (in two variables) is a polynomial $q(x,y)$ of the form $a+bx+cy+dx^2+exy+fy^2$ for some real scalars a, b, c, d, e, f. Let V be the vector space of those quadratic polynomials which have $q(1,1) = q(1,-1) = q(-1,1) = q(-1,-1) = 0$.

(a) (9 points) Find a basis for V.

The condition $q(1,1) = q(1,-1) = q(-1,1) = q(-1,-1) = 0$ unpacks to four linear equations:

$$\begin{aligned} a+b+c+d+e+f &= 0 \\ a-b+c+d-e+f &= 0 \\ a+b-c+d-e+f &= 0 \\ a-b-c+d+e+f &= 0 \end{aligned}$$

Subtracting the first rows from the others gives

$$\begin{array}{rcl} a+ b + c +d+ e +f &=& 0 \\ -2b \qquad\quad -2e &=& 0 \\ -2c \quad -2e &=& 0 \\ -2b-2c \qquad\quad &=& 0 \end{array}$$

We could keep row reducing at this point, but it is easier to directly notice that the bottom three equations give $b = -e = c = -b$, so $b = -b$ and we deduce that $b = c = e = 0$. Thus, our equations simplify to

$$a + d + f = 0 \qquad b = c = e = 0.$$

A basis of solutions is $(a,b,c,d,e,f) = (1,0,0,-1,0,0)$ and $(1,0,0,0,0,-1)$. The corresponding polynomials are $1 - x^2$ and $1 - y^2$.

(b) (5 points) Express the quadratics $x^2 - y^2$, $x^2 + y^2 - 2$ and $x^2 - 1$ in your basis.

We have

$$\begin{array}{lcll} x^2 - y^2 &=& -(1 - x^2) &+ \ (1 - y^2) \\ x^2 + y^2 - 2 &=& -(1 - x^2) &- \ (1 - y^2) \\ x^2 - 1 &=& -(1 - x^2) & \end{array}$$

Of course, the answer to this part will depend on the answer to the previous part.

(c) (5 points) Are the three quadratics $x^2 - y^2$, $x^2 + y^2 - 2$ and $x^2 - 1$ linearly independent, or linearly dependent? Justify your answer.

They are linearly dependent. The vector space of solutions has dimension 2, so any three vectors must be linearly dependent.

Question 3 (10 points)

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and let L be the plane $\mathrm{Span}(\vec{u}, \vec{v})$ in $\mathbb{R}^3$.

(a) (4 points) The vector $\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ is in L. Write $\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ in the coordinates of the basis $(\vec{u}, \vec{v})$.

We have

$$\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \vec{u} - 2\vec{v}$$

so the answer is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(b) (6 points) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ -8x \end{bmatrix}$; the transformation A takes L to L. Write the restriction of A to L in the basis $(\vec{u}, \vec{v})$ of L.

We have

$$\begin{aligned} A\vec{u} &= \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix} &= 2\vec{u} - 4\vec{v} \\ A\vec{v} &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} &= \vec{u} \end{aligned}$$

so the answer is $\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix}$.

Question 4 (16 points)
Let V be a vector space and let $A : V \to V$ be a linear map. Please prove or provide a counterexample to the following claims:

(a) (4 points) If $B : V \to V$ is invertible, then $\text{Ker}(BA) = \text{Ker}(A)$.
This is true. If $A\vec{v} = \vec{0}$ then $BA\vec{v} = B\vec{0} = \vec{0}$. Conversely, if $BA\vec{v} = \vec{0}$ then $A\vec{v} = B^{-1}BA\vec{v} = B^{-1}\vec{0} = \vec{0}$.

(b) (4 points) If $B : V \to V$ is invertible, then $\text{Ker}(AB) = \Im(A)$.
This is false. A counter-example is to take $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $AB = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. So $\text{Ker}(A) = \left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\right]$ and $\text{Ker}(AB) = \left[\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right]$.

(c) (4 points) If A^2 is invertible, then A is invertible.
This is true. If $SA^2 = \text{Id}$ then $(SA)A = \text{Id}$, so SA is the inverse to A. Since the matrices are square, we only need to check one of the two identities $(SA)A = \text{Id}$ and $A(SA) = \text{Id}$.

(d) (4 points) If $A^2 = 0$, then $A = 0$.
This is false. A counter-example is $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Question 5 (10 points)

Let V be a finite dimensional vector space with basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Let $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ be another vector in V. Prove that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{w}$ is a basis of V if and only if $c_n \neq 0$.

First, suppose that $c_n = 0$. Then

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{n-1}\vec{v}_{n-1}.$$

So $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{w}$ are linearly dependent and are not a basis.

Now, suppose that $c_n \neq 0$. We will show that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{w}$ are linearly independent, and span V. We could also check just one of these, and then note that this is a list of n elements and $\dim V = n$.

Proof of linear independence: Suppose that $a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1} + b\vec{w} = \vec{0}$. We rewrite this as

$$a_1\vec{v}_1 + \cdots + a_{n-1}\vec{v}_{n-1} + b(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = \vec{0}.$$

$$(a_1 + bc_1)\vec{v}_1 + (a_2 + bc_2)\vec{v}_2 + \cdots (a_{n-1} + bc_{n-1})\vec{v}_{n-1} + bc_n\vec{v}_n = \vec{0}.$$

Using the linear independence of the $\vec{v}_i$, we have

$$a_1 + bc_1 = a_2 + bc_2 = \cdots = a_{n-1} + bc_{n-1} = bc_n = 0.$$

Since $c_n \neq 0$, we deduce that $b = 0$ and thus $a_1 = a_2 = \cdots = a_{n-1} = 0$.

Proof of spanning: We have

$$\vec{v}_n = \tfrac{1}{c_n}\left(\vec{w} - c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_{n-1}\vec{v}_{n-1}\right).$$

So $\vec{v}_n$ is in $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{w})$. Since $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V, this shows that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}, \vec{w}$ spans V as well.

Question 6 (10 points)

Let V be a finite dimensional vector space and let A and B be subspaces. Show that, if $\dim A + \dim B > \dim V$, then $A \cap B$ contains a nonzero vector.

Here is one approach. Let $\alpha_1, \alpha_2, \ldots, \alpha_a$ be a basis of A and let $\beta_1, \beta_2, \ldots, \beta_b$ be a basis of B. Since $a + b > \dim V$, there is a linear relation:

$$c_1\alpha_1 + \cdots + c_a\alpha_a + d_1\beta_1 + \cdots + d_b\beta_b = \vec{0}$$

whose coefficients are not all 0. So

$$c_1\alpha_1 + \cdots + c_a\alpha_a = -d_1\beta_1 - \cdots - d_b\beta_b.$$

The sum $c_1\alpha_1 + \cdots + c_a\alpha_a$ is clearly in A, and the sum $-d_1\beta_1 - \cdots - d_b\beta_b$ is clearly in B, so this is a vector in $A \cap B$. Using the linear independence of $\alpha_1, \alpha_2, \ldots, \alpha_a$, and of $\alpha_1, \alpha_2, \ldots, \alpha_a$, it is not zero.

Question 7 (20 points)

Let $\mathbb{R}^\infty$ be the vector space of infinite sequences $(a_1, a_2, a_3, \ldots)$ of real numbers. Let $S : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the linear transformation

$$S\big((a_1, a_2, a_3, \ldots)\big) = (a_2, a_3, a_4, \ldots).$$

(a) (4 points) What is the image of S?

Every vector is in the image of S, since $(a_2, a_3, a_4, \ldots)$ is $S\big((0, a_2, a_3, \ldots)\big)$.

(b) (4 points) What is the kernel of S?

The kernel of S is vectors of the form $(a, 0, 0, 0, \ldots)$.

(c) (6 points) Is there a linear transformation T such that $ST = \mathrm{Id}$? Explain why or why not.

Yes. One such choice is $T\big((a_1, a_2, \ldots,)\big) = (0, a_1, a_2, \ldots)$.

(d) (6 points) Is there a linear transformation T such that $TS = \mathrm{Id}$?

No. We have $S\big((1, 0, 0, 0, \ldots)\big) = S\big((0, 0, 0, 0, \ldots)\big) = (0, 0, 0, \ldots)$ so, if there were such a T, we would have both $T\big((0, 0, 0, 0, \ldots)\big) = (0, 0, 0, \ldots)$ and $T\big((0, 0, 0, 0, \ldots)\big) = (1, 0, 0, \ldots)$, a contradiction.