Question 1

Let

$$B = \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix}$$

and suppose $\det B = 17$. Compute the following quantities, and show how you did so:

(a) $\det \begin{bmatrix} 2p & 2q & 2r \\ 2s & 2t & 2u \\ 2v & 2w & 2x \end{bmatrix}$.

Factoring out a 2 from each row (or each column), we have

$$\det \begin{bmatrix} 2p & 2q & 2r \\ 2s & 2t & 2u \\ 2v & 2w & 2x \end{bmatrix} = 2^3 \det \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} = 2^3 \times 17 = 136.$$

(b) $\det \begin{bmatrix} s & t & u \\ v & w & x \\ p & q & r \end{bmatrix}$.

We perform row swaps to return to the original matrix:

$$\det \begin{bmatrix} s & t & u \\ v & w & x \\ p & q & r \end{bmatrix} = -\det \begin{bmatrix} s & t & u \\ p & q & r \\ v & w & x \end{bmatrix} = \det \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} = 17.$$

(c) $\det \begin{bmatrix} p & q & r \\ 2p+3s & 2q+3t & 2r+3u \\ 4p+5v & 4q+5w & 4r+5x \end{bmatrix}$.

We perform row operations to give:

$$\det \begin{bmatrix} p & q & r \\ 2p+3s & 2q+3t & 2r+3u \\ 4p+5v & 4q+5w & 4r+5x \end{bmatrix} = \det \begin{bmatrix} p & q & r \\ 3s & 3t & 3u \\ 5v & 5w & 5x \end{bmatrix} = 3 \times 5 \times \det \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} = 3 \times 5 \times 17 = 255.$$

Question 2

Let V be the vector space of functions of the form $f(x)e^{-x^2/2}$ where f is a polynomial of degree $\leq n$. For $f \in V$, put $D(f) = \frac{d^2 f}{(dx)^2} - x^2 f$.

(a) Show that D maps V to V.

For any polynomial $f(x)$, we have

$$\begin{aligned} D\left(f(x)e^{-x^2/2}\right) &= \tfrac{d^2}{(dx)^2}\left(f(x)e^{-x^2/2}\right) - x^2 f(x)e^{-x^2/2} \\ &= \tfrac{d}{dx}\left(f'(x)e^{-x^2/2} - xf(x)e^{-x^2/2}\right) - x^2 f(x)e^{-x^2/2} \\ &= \left(f''(x)e^{-x^2/2} - xf'(x)e^{-x^2/2} - f(x)e^{-x^2/2} - xf'(x)e^{-x^2/2} + x^2 f(x)e^{-x^2/2}\right) - x^2 f(x)e^{-x^2/2} \\ &= \left(-f''(x) - 2xf'(x) - f(x)\right)e^{-x^2/2}. \end{aligned}$$

If f has degree $\leq n$, then f'' has degree $\leq n-2$, $2xf'(x)$ has degree $\leq n$ and f has degree $\leq n$, so $-f''(x) - 2xf'(x) - f(x)$ has degree $\leq n$.

Note that it is very important that the $x^2 f(x)$ terms cancelled; otherwise, this wouldn't be true.

(b) Compute the eigenvalues of D acting on V.

Using the above formula, we see that $D(x^n e^{-x^2/2}) = \left(-n(n-1)x^{n-2} - (2n+1)x^n\right) e^{-x^2/2}$. Thus, in the basis x^n, x^{n-1}, …, x^0, the matrix of D is

$$\begin{bmatrix} -(2n+1) & & & & & & \\ 0 & -(2n-1) & & & & & \\ -n(n-1) & 0 & -(2n-3) & & & & \\ & -(n-1)(n-2) & 0 & -(2n-5) & & & \\ & & -(n-2)(n-3) & 0 & -(2n-7) & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 2(2-1) & 0 & -1 \end{bmatrix}.$$

(All blank entries are 0.) So the matrix is lower triangular with diagonal entries $-(2n+1)$, $(-2n-1)$, $-(2n-3)$, …, -1, and the eigenvalues are $-(2n+1)$, $(-2n-1)$, $-(2n-3)$, …, -1.

Question 3

Let

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $(x-1)(x^2+1)$.

(a) Find a real eigenvector of A.

Since the characteristic polynomial is $(x-1)(x^2+1)$, we know that the only real eigenvalue is 1. We compute the kernel of $A - \mathrm{Id}_3$:

$$\mathrm{Ker}(A - \mathrm{Id}_3) = \mathrm{Ker} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 2 \\ -1 & 2 & -2 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

(b) Find a two dimensional subspace L of $\mathbb{R}^3$ such that A maps L to L.

Guided by the degree two factor of the characteristic polynomial, we compute $\mathrm{Ker}(A^2 + \mathrm{Id}_3)$.

$$\mathrm{Ker}(A^2 + \mathrm{Id}_3) = \mathrm{Ker} \begin{bmatrix} -4 & 12 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \mathbb{R} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Question 4

Let A be a real square $k \times k$ matrix which is diagonalizable with eigenvalues 2 and 3. Let $\vec{v}$ be a k-dimensional real vector. Show that the sequence of vectors $\vec{v}$, $A\vec{v}$, $A^2\vec{v}$, $A^3\vec{v}$, … obeys

$$A^{n+2}\vec{v} = 5A^{n+1}\vec{v} - 6A^n\vec{v}$$

for all positive integers n.

Since A is diagonalizable with eigenvalues 2 and 3, we have $(A - 2\mathrm{Id})(A - 3\mathrm{Id}) = 0$ or, in other words, $A^2 - 5A + \mathrm{Id} = 0$. Multiplying by A^n and rearranging, we have $A^{n+2} = 5A^{n+1} - 6A^n$, so $A^{n+2}\vec{v} = 5A^{n+1}\vec{v} - 6A^n\vec{v}$.

Question 5

Let V be a two dimensional vector space over the field of rational numbers. **Note that our field is the rational numbers, not the reals.** Let $T : V \to V$ be a linear transformation obeying $T^2 = 2\mathrm{Id}$. Let $\vec{v} \in V$ be a nonzero vector.

(a) Prove that $\vec{v}$ and $T\vec{v}$ are linearly independent.

Suppose, to the contrary, that $T\vec{v} = c\vec{v}$. Then $T^2\vec{v} = c^2\vec{v}$. But also $T^2\vec{v} = 2\vec{v}$, so $c^2 = 2$. But 2 has no rational square root, a contradiction.

(b) Write the linear operator T in the basis $\vec{v}$, $T\vec{v}$.
We have $T\vec{v} = T\vec{v}$ and $T(T\vec{v}) = T^2\vec{v} = 2\vec{v}$, so the matrix is $\left[\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right]$.

Question 6

Let A be a 3×3 real matrix with eigenvalues 1, 2 and 3. Let L be the set of 3×3 matrices B which obey the condition $AB = BA$.

(a) Show that L is 3-dimensional.
We work in a basis where A is diagonal, so

$$A = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}.$$

We write out the condition $AB = BA$:

$$AB = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 2B_{21} & 2B_{22} & 2B_{23} \\ 3B_{31} & 3B_{32} & 3B_{33} \end{bmatrix}$$

$$BA = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} = \begin{bmatrix} B_{11} & 2B_{12} & 3B_{13} \\ B_{21} & 2B_{22} & 3B_{23} \\ B_{31} & 2B_{32} & 3B_{33} \end{bmatrix}.$$

Equating the two sides, for $i \neq j$, we have $iB_{ij} = jB_{ij}$ so $B_{ij} = 0$. Thus, A and B commute if and only if B is diagonal (in the same basis where A is diagonal).

(b) Show that a basis of L is Id, A, A^2.
(There are several ways to solve this question; this is the most conceptual.) First, we check that these matrices are in L. Indeed, $A\mathrm{Id} = A = \mathrm{Id}A$, $AA = A^2 = AA$ and $AA^2 = A^3 = A^2A$.
We have already checked that L is three dimensional, so it remains to show that Id, A and A^2 are linearly independent. Suppose, instead, that we had a linear relation $c_2A^2 + c_1A + c_0\mathrm{Id} = 0$. Then A is a zero of the polynomial $c_2x^2 + c_1x + c_0$. But the minimal polynomial of A is $(x-1)(x-2)(x-3)$, so A is not a zero of any polynomial of degree ≤ 2.

(c) Express A^3 as a linear combination of Id, A and A^2, and prove your answer is correct.
The easiest way to do this is to use the minimal polynomial: We have $(A - \mathrm{Id})(A - 2\mathrm{Id})(A - 3\mathrm{Id}) = 0$, or $A^3 - 6A^2 + 11A - 6\mathrm{Id} = 0$, so $A^3 = 6A^2 - 11A + 6\mathrm{Id}$.

Question 7

Let V be a finite dimensional vector space and let V^* be the dual vector space. Let α, β and γ be three elements of V^*.

(a) Show that α, β and γ are linearly independent **if and only if** there are vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ in V obeying the equations:

$$\begin{array}{lllllllll} \alpha(\vec{u}) & = & 1 & \alpha(\vec{v}) & = & 0 & \alpha(\vec{w}) & = & 0 \\ \beta(\vec{u}) & = & 0 & \beta(\vec{v}) & = & 1 & \beta(\vec{w}) & = & 0 \\ \gamma(\vec{u}) & = & 0 & \gamma(\vec{v}) & = & 0 & \gamma(\vec{w}) & = & 1 \end{array}$$

First suppose that such vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ exist and suppose, for the sake of contradiction, that we have $a\alpha + b\beta + c\gamma = 0$. Then $0 = (a\alpha + b\beta + c\gamma)(\vec{u}) = a\alpha(\vec{u}) + b\beta(\vec{u}) + c\gamma(\vec{u}) = a \cdot 1 + b \cdot 0 + c \cdot 0 = a$ and similarly $0 = b$ and $0 = c$.
In the reverse order, suppose that α, β and γ are linearly independent. Complete α, β, γ to a basis α, β, γ, δ_4, δ_5, …, δ_n of V^*. Let $\vec{u}$, $\vec{v}$, $\vec{w}$, e_4, e_5, …, e_n be the dual basis of V.

(b) Recall that $\alpha \wedge \beta \wedge \gamma$ is defined to be the multilinear form

$$(\alpha \wedge \beta \wedge \gamma)(\vec{x}, \vec{y}, \vec{z}) = \alpha(\vec{x})\beta(\vec{y})\gamma(\vec{z}) - \alpha(\vec{x})\beta(\vec{z})\gamma(\vec{y}) - \alpha(\vec{y})\beta(\vec{x})\gamma(\vec{z}) + \alpha(\vec{y})\beta(\vec{z})\gamma(\vec{x}) + \alpha(\vec{z})\beta(\vec{x})\gamma(\vec{y}) - \alpha(\vec{z})\beta(\vec{y})\gamma(\vec{x}).$$

Show that α, β and γ are linearly independent **if and only if** $\alpha \wedge \beta \wedge \gamma \neq 0$.

First, suppose that α, β and γ are linearly independent and take $\vec{u}$, $\vec{v}$ and $\vec{w}$ as above. Then

$$(\alpha \wedge \beta \wedge \gamma)(\vec{u}, \vec{v}, \vec{w}) = 1 \cdot 1 \cdot 1 + 0 + 0 + 0 + 0 + 0 = 1.$$

So $\alpha \wedge \beta \wedge \gamma \neq 0$.

Conversely, suppose that one of α, β, γ is in the span of the other two: Say, without loss of generality, that $\gamma = a\alpha + b\beta$. Then

$$\begin{aligned} \alpha \wedge \beta \wedge \gamma = \alpha \wedge \beta \wedge (a\alpha + b\beta) &= a(\alpha \wedge \beta \wedge \alpha) + b(\alpha \wedge \beta \wedge \beta) \\ &= -a(\alpha \wedge \alpha \wedge \beta) + b(\alpha \wedge \beta \wedge \beta) = -a(0 \wedge \beta) + b(\alpha \wedge 0) = 0. \end{aligned}$$

Question 8

Let U and V be $n \times n$ invertible matrices which obey $UV = -VU$. Show that, if λ is an eigenvalue of U, then $-\lambda$ is also an eigenvalue of U.

We rewrite $UV = -VU$ as $U = -VUV^{-1}$. So U and $-U$ are similar and have the same eigenvalues. If λ is an eigenvalue of U, then $-\lambda$ is an eigenvalue of $-U$ so, by the similarity, we deduce that $-\lambda$ is an eigenvalue of U as well.