Problem Set One: Due Thursday, January 13 at 11:59 PM

See course website for homework policies.

Reading Read Chapter 1 looking for material you haven't seen before. The field material can be skimmed for now. Then fill out the poll at `https://forms.gle/jxiLMBnrKijx9nJK8` .

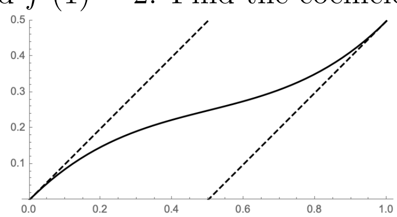
Problem 1. We list some basic nutritional information:

- One cup of flour contains 80 grams of carbohydrate, 16 grams of protein and no fat.
- One egg contains no carbohydrate, 6 grams of protein and 5 grams of fat.
- One tablespoon of butter contains no carbohydrate or protein and 11 grams of fat.
- One cup of sugar contains 192 grams of carbohydrate and no protein or fat.

Compute the matrix R such that:

$$R \begin{bmatrix} \text{Flour} \\ \text{Eggs} \\ \text{Butter} \\ \text{Sugar} \end{bmatrix} = \begin{bmatrix} \text{Carbohydrate} \\ \text{Protein} \\ \text{Fat} \end{bmatrix}.$$

Problem 2. The picture below depicts a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ with $f(0) = 0$, $f'(0) = 2$, $f(1) = 1$ and $f'(1) = 2$. Find the coefficients of $f(x)$.



Problem 3. Let A be an $\ell \times m$ matrix and let B be an $m \times n$ matrix.

(1) Prove or disprove: If $\text{Ker}(A) = \{\vec{0}\}$ and $\text{Ker}(B) = \{\vec{0}\}$ then $\text{Ker}(AB) = \{\vec{0}\}$.
(2) Prove or disprove: If $\text{Ker}(AB) = \{\vec{0}\}$ then $\text{Ker}(A) = \{\vec{0}\}$ and $\text{Ker}(B) = \{\vec{0}\}$.
(3) Prove or disprove: If $\text{Im}(A) = \mathbb{R}^\ell$ and $\text{Im}(B) = \mathbb{R}^m$ then $\text{Im}(AB) = \mathbb{R}^\ell$.
(4) Prove or disprove: If $\text{Im}(AB) = \mathbb{R}^\ell$ then $\text{Im}(A) = \mathbb{R}^\ell$ and $\text{Im}(B) = \mathbb{R}^m$.

Problem 4. Let A be an $m \times n$ matrix and let V be an $n \times n$ invertible matrix.

(1) Prove that $\text{Im}(AV) = \text{Im}(A)$.
(2) Prove that $\text{Ker}(AV) = V^{-1}\,\text{Ker}(A)$. To spell this out more explicitly, show that $\text{Ker}(AV)$ is $\{V^{-1}\vec{x} : \vec{x} \text{ in } \text{Ker}(A)\}$.

Problem 5. (**Challenge**) We described three row operations that we can perform on a matrix:

(R1) Adding a multiple of row i to row j.
(R2) Switching rows i and j.
(R3) Multiplying row i by a nonzero scalar c.

This problem investigates what we can do using by just doing $(R1)$ many times.

(1) Let $\vec{x}$ and $\vec{y}$ be two rows of A. Show that, by using $(R1)$ three times, we can change these rows into $-\vec{y}$ and $\vec{x}$.
(2) Let $\vec{x}$ and $\vec{y}$ be two rows of A and let c be a nonzero scalar. Show that, by using $(R1)$ four times, we can change these rows into $c\vec{x}$ and $c^{-1}\vec{y}$.