Problem Set Ten: Due Thursday, April 7 at 11:59 PM

See course website for homework policies.

Reading Read 8.1-8.4.

Textbook problems Please solve **8.2.1**, **8.2.2**, **8.2.12**, **8.4.4**, **8.4.8**.

Problem 1. In this problem, we will prove the following result: Let A be a square matrix and suppose that the characteristic polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) = \prod (x - \lambda_i)^{n_i}$. Then there is a basis in which A is upper triangular. (Of course, if all the eigenvalues are already distinct, we know that A is diagonalizable.)

(1) Let V be an m-dimensional vector space and let $C : V \to V$ be a linear transformation with $C^m = 0$. Show that V has a basis $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_m$ such that $C(\vec{v}_i) \in \operatorname{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1})$. Conclude that, in this basis, C is upper triangular with 0's on the diagonal.
(2) Let V be an m-dimensional vector space, let λ be a scalar and let $B : V \to V$ be a linear transformation with $\chi_B(x) = (x - \lambda)^m$. Show that there is a basis for V in which B is upper triangular with λ's on the diagonal.
(3) Let V be an m-dimensional vector space, let $A : V \to V$ be a linear transformation and suppose that the minimal polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) = \prod (x - \lambda_i)^{n_i}$. Show that there is a basis for V where A is upper triangular with the λ_i on the diagonal.

Problem 2. Let F be a field and let $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ be an ***irreducible*** polynomial with coefficients in F.

(1) Let V be an n-dimensional vector space and let $A : V \to V$ be a linear transformation with $\chi_A(x) = f(x)$. Let $\vec{v}$ be any nonzero vector in V. Show that $\vec{v}$, $A\vec{v}$, $\dots$, $A^{n-1}\vec{v}$ is a basis of V.
(2) Let A and V be as in the previous part. Write the matrix of A in the basis $\vec{v}$, $A\vec{v}$, $\dots$, $A^{n-1}\vec{v}$.

Problem 3. Let V be the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V by

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx.$$

(1) Show that the following list of functions is orthonormal: $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}} \sin(nx)$ for $n \geq 1$, and $\frac{1}{\sqrt{\pi}} \cos(nx)$ for $n \geq 1$.
(2) Let $f(x) = x$. Find the function in $\operatorname{Span}(\sin x, \sin(2x), \sin(3x))$ which is closest to the function $f(x)$. As an incentive, here is a plot of $y = x$ and of $y = a_1 \sin(x) + a_2 \sin(2x) + a_3 \sin(3x)$ for optimally chosen a_1, a_2, a_3.