Problem Set Eleven: Due Teusday, April 19 at 11:59 PM

See course website for homework policies. **This is the final problem set!**

Reading Read 8.5.

Textbook problems Please solve **8.5.1**, **8.5.3**, **8.5.6**, **8.5.9**, **8.5.10**, **8.5.11**

Problem 1. Let V be the vector space of smooth (meaning infinitely differentiable) functions $[0, 2\pi] \to \mathbb{R}$ which obey $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$. Define an inner product on V by

$$\langle f(x), g(x) \rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Define the linear operator $L : V \to V$ by $L(f) = \frac{d^2}{(dx)^2} f$. Show that L is selfadjoint, meaning that $\langle L(f), g \rangle = \langle f, L(g) \rangle$.

Problem 2. Let A be a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. In this problem, we will show that A has a singular value decomposition, meaning that we can find two orthonormal bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ for $\mathbb{R}^n$ such that $A\vec{u}_i$ is a scalar multiple of $\vec{v}_i$ for each $1 \leq i \leq n$.

(1) Consider the function $|A\vec{x}|$ on the unit sphere $\{\vec{x} | \langle \vec{x}, \vec{x} \rangle = 1\}$. Let $\vec{u}$ be the vector on the unit sphere where $|A\vec{u}|$ is maximized. (You may assume such a vector exists, if you don't have the analysis background to know that a continuous function on a compact set always has a maximum.) Define $\vec{v} = A\vec{u}$. Show that A takes $\vec{u}^{\perp}$ to $\vec{v}^{\perp}$.

(2) Show (induct on n) there there is a pair of orthonormal bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ for $\mathbb{R}^n$ such that $A\vec{u}_i$ is a scalar multiple of $\vec{v}_i$ for each $1 \leq i \leq n$.