Problem Set Two: Due Thursday, January 20 at 11:59 PM

See course website for homework policies.

Reading Read 1.1 and 2.1-2.3. Then fill out the poll at `https://forms.gle/GQnxU7gxfCvk7QFWA`

Problem 1. Find linear polynomials $at+b$ and $ct+d$ such that

$$(at+b)(t^2+1)+(ct+d)(t^2+t+1)=1.$$

Problem 2. Find a nonzero vector which is both in Image $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and in Image $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

Problem 3. Let F be a field (see Section 1.1) in your textbook. Prove the following from the axioms of a field; you may also use the results that $0x = x0 = 0$ and $(-1)x = x(-1) = -x$.

(1) For any x in F, we have $x^2 - 1 = (x+1)(x-1)$.
(2) For any elements x and y in F, if $xy = 0$ then either $x = 0$ or $y = 0$.
(3) For any x in F, if $x^2 = 1$ then $x = 1$ or $x = -1$.

Problem 4. Let A be an $\ell \times m$ matrix and let B be an $m \times n$ matrix.

(1) Suppose that $\mathrm{Ker}(AB) = 0$ and $\mathrm{Image}(B) = \mathbb{R}^m$. Show that $\mathrm{Ker}(A) = \{\vec{0}\}$.
(2) Suppose that $\mathrm{Image}(AB) = \mathbb{R}^\ell$ and $\mathrm{Ker}(A) = \{\vec{0}\}$. Show that $\mathrm{Image}(B) = \mathbb{R}^m$.

Problem 5. Let X be a set and let F be a field. Let F^X be the vector space of all functions $f : X \to F$. (See Example 3 in Section 2.1 of your textbook.) Let F^X_{finite} be the set of functions $f : X \to F$ such that $\{x \in X : f(x) \neq 0\}$ is finite. Show that F^X_{finite} is a subspace of F^X.

Problem 6. Let T be the set of functions $\mathbb{R} \to \mathbb{R}$ which are of the form $a \cos t + b \sin t$. For $x + iy$ in $\mathbb{C}$ and $f(t)$ in T, define $(x+iy) * f(t) = xf(t) + y\frac{df}{dt}$. Show that T is a $\mathbb{C}$-vector space with respect to this scalar multiplication, and the usual addition.