Problem Set Four: Due Thursday, February 3 at 11:59 PM

See course website for homework policies.

Reading Read 3.1-3.4 Then fill out the poll at `https://forms.gle/NZByXgpwJFyEoHW87` .

Textbook problems Please solve problems **2.4.1**, **2.4.3**, **2.4.5**, **2.4.6**, **2.6.3**, and **2.6.6**.

Problem 1. Let $X = \{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} : x, y \in \mathbb{R}\}$ and let $Y = \{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R}\}$. Show that $\mathbb{R}^3 = X \oplus Y$.

Problem 2. Consider the following subspaces of $\mathbb{R}[x]$:

$$\begin{array}{rcl} C & = & \{\text{constant polynomials}\} \\ L & = & \{\text{polynomials of degree} \leq 1\} \\ P & = & \{f(x) : f(0) = 0\} \\ Q & = & \{f(x) : f(0) = 0 \text{ and } f(1) = 0\} \end{array} .$$

Prove or disprove each of the following statements:

(1) $\mathbb{R}[x] = C \oplus P$.
(2) $\mathbb{R}[x] = C \oplus Q$.
(3) $\mathbb{R}[x] = L \oplus P$.
(4) $\mathbb{R}[x] = L \oplus Q$.

Problem 3. Let V be a finite dimensional vector space and let $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_k$ be a set of linearly independent vectors in V. Show that there exist vectors $\vec{v}_{k+1}$, ..., $\vec{v}_n$ in V such that $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_k$, $\vec{v}_{k+1}$, ..., $\vec{v}_n$ is a basis of V.

Problem 4. Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation. Define $I_j = \text{Image}(T^j)$ and $K_j = \text{Ker}(T^j)$.

(1) Show that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$.
(2) Show that there is some integer N for which $I_N = I_{N+1} = I_{N+2} = \cdots$ and $K_N = K_{N+1} = K_{N+2} = \cdots$.
(3) Show that T maps I_N to I_N and the map $T : I_N \to I_N$ is invertible.
(4) Show that $V = I_N \oplus K_N$.