Problem Set Six: Due Thursday, February 24 at 11:59 PM

See course website for homework policies.

Reading Read 5.2, 5.3 and 5.4.

Textbook problems Please solve problems **5.2.3**, **5.2.4**, **5.2.9**, **5.2.10**, **5.3.7**.

Problem 1. Let V and W be vector spaces and let $A : V \to W$ be a linear transformation.

(1) Show that $\mathrm{Ker}(A^*) = \mathrm{Im}(A)^\perp$.
(2) Show that, if V and W are finite dimensional, we also have $\mathrm{Im}(A^*) = \mathrm{Ker}(A)^\perp$.

Problem 2. Let V be a vector space over $\mathbb{R}$, and let $A : V \times V \times V \longrightarrow \mathbb{R}$ be an alternating multilinear form. Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be three vectors in V with $A(\vec{x}, \vec{y}, \vec{z}) = 17$. Compute the following, directly using the axioms of an alternating form:

(1) $A(\vec{y}, \vec{z}, \vec{x})$.
(2) $A(\vec{x}, 2\vec{x} + 3\vec{y}, 4\vec{x} + 5\vec{y} + 6\vec{z})$.
(3) $A(\vec{x} + 2\vec{y} + 3\vec{z}, 4\vec{x} + 5\vec{y}, 6\vec{x})$.
(4) $A(2\vec{x} + \vec{y}, \vec{x} + 2\vec{y}, \vec{z})$.

Problem 3. Let V be a vector space of dimension n over a field F. Let $A : V \times V \times V \longrightarrow F$ be a multilinear form. We will say that A is ***symmetric*** if, for all vectors $\vec{u}$, $\vec{v}$, $\vec{w} \in V$, we have

$$A(\vec{u}, \vec{v}, \vec{w}) = A(\vec{u}, \vec{w}, \vec{v}) = A(\vec{v}, \vec{u}, \vec{w}) = A(\vec{v}, \vec{w}, \vec{u}) = A(\vec{w}, \vec{u}, \vec{v}) = A(\vec{w}, \vec{v}, \vec{u}).$$

What is the dimension of the vector space of symmetric bilinear forms $A : V \times V \times V \longrightarrow F$?

Problem 4. Let H be the vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(0) = f(1) = 0$. For $f(x)$ and $g(x)$ in H, define

$$\langle f, g \rangle = \int_0^1 f(x) g'(x) dx.$$

Show that $\langle\ ,\ \rangle$ is an alternating bilinear form $H \times H \to \mathbb{R}$. (You need to check both that it is bilinear and that it is alternating.)