Problem Set Seven: Due Thursday, March 10 at 11:59 PM

See course website for homework policies.

Reading Read 5.6 and 5.7.

Problem 1. Recall that the cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3-y_2x_3 \\ x_3y_1-y_3x_1 \\ x_1y_2-y_1x_2 \end{bmatrix}.$$

For any vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ in $\mathbb{R}^3$, define $B_{\vec{c}}(\vec{x}, \vec{y}) := \vec{c} \cdot (\vec{x} \times \vec{y})$.

(1) Show that, for any vector $\vec{c} \in \mathbb{R}^3$, the function $B_{\vec{c}}(\ , \ )$ is an alternating bilinear form.
(2) Let $B(\ , \ )$ be any alternating bilinear form on $\mathbb{R}^3$. Show that there is a unique vector $\vec{c} \in \mathbb{R}^3$ such that $B(\ , \ )$ is $B_{\vec{c}}(\ , \ )$.

Problem 2. Let V be an n dimensional vector space over a field F. Let $e_1, e_2, \ldots, e_n$ be one basis for V and let $f_1, f_2, \ldots, f_n$ be another basis. Let S be the matrix defined by $f_j = \sum_i S_{ij} e_i$.

(1) Let $T : V \to V$ be a linear map and define the matrices X and Y by $T(e_j) = \sum_i X_{ij} e_i$ and $T(f_j) = \sum_i Y_{ij} f_i$. Give a formula for Y in terms of X and S.
(2) Show that $\det X = \det Y$.
(3) Let $B : V \times V \longrightarrow F$ be a bilinear form and define the matrices P and Q by $B(e_i, e_j) = P_{ij}$ and $B(f_i, f_j) = Q_{ij}$. Give a formula for Q in terms of P and S.
(4) Show that there is a nonzero element $s \in F$ with $\det P = s^2 \det Q$.

Problem 3. Let V be an n-dimensional vector space over a field F and let $B : V \times V \to F$ be an alternating bilinear form. In this problem, we will show that there is some integer r such that there is a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2, \ldots, \vec{x}_r, \vec{y}_r, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-2r}$ of V such that $B(\vec{x}_i, \vec{y}_i) = -B(\vec{y}_i, \vec{x}_i) = 1$ and all other pairings between the basis vectors are 0. This proof is by induction on n.

(1) Do the base cases $n = 1$ and $n = 2$.
(2) Explain why we are done if $B(\vec{v}, \vec{w}) = 0$ for all vectors $\vec{v}$ and $\vec{w}$ in V.

From now on, assume that $n > 2$ and that $B(\vec{v}, \vec{w})$ is not always 0. Choose two vectors $\vec{x}$, $\vec{y}$ with $B(\vec{x}, \vec{y}) = 1$. Set $V' = \{\vec{v} : B(\vec{x}, \vec{v}) = B(\vec{y}, \vec{v}) = 0\}$.

(3) Show that $V = \mathrm{Span}(\vec{x}, \vec{y}) \oplus V'$.
(4) By induction, V' has a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2, \ldots, \vec{x}_r, \vec{y}_r, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{n-2-2r}$ as required. Explain how to finish the proof from here.
(5) We conclude with an example. Consider the alternating bilinear form

$$B((u_1, u_2, u_3, u_4), \ (v_1, v_2, v_3, v_4)) = \sum_{1 \leq i < j \leq 4} (u_i v_j - u_j v_i)$$

on $\mathbb{R}^4$. Find a basis $\vec{x}_1, \vec{y}_1, \vec{x}_2, \vec{y}_2$ as above.