Problem Set Eight: Due Thursday, March 17 at 11:59 PM

See course website for homework policies.

**Reading** Read 6.1-6.4.

**Textbook problems** Please solve **6.2.3**, **6.2.4**, **6.2.10**, **6.4.2**, **6.4.5**, **6.4.7**.

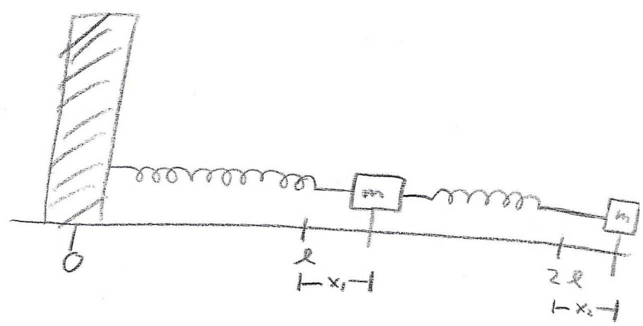
**Problem 1.** Let $V$ be a finite dimensional vector space, let $A : V \to V$ be a linear transformation and suppose that $U$ is a subspace of $V$ such that $AU \subseteq U$.

(1) Show that there is a basis of $V$ in which $A$ takes the form $\left[\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right]$.
(2) Show that there is a basis of $U$ such that the restriction $A|_U$ given by the matrix $P$.
(3) Show that there is a linear map $\bar{A} : V/U \longrightarrow V/U$ defined by $\bar{A}(v + U) = A(V) + U$. (In other words, show that, if $v_1 + U = v_2 + U$, then $\bar{A}(v_1) + U = \bar{A}(v_2) + U$ and this function $V/U \longrightarrow V/U$ is linear.)
(4) Show that there is a basis for $V/U$ where $\bar{A}$ is given by the matrix $R$.

**Problem 2.** For a polynomial $f$ with real coefficients, define $D(f) = xf' + f''$. For each positive integer $n$, show that there is a polynomial of degree $\leq n$ such that $D(f) = nf$. (Hint: What does this have to do with eigenvalues?)

**Problem 3.** In this problem, we will discuss the relevance of eigenvalues to oscillations of mechanical systems. If you hate physics, skip to the differential equation below.

Consider the physical system drawn below: There is a frictionless track with two masses resting on it, each of length $m$. There is a spring from the first mass to an anchor at the origin, and a spring between the two masses, each of which have rest length $\ell$ and spring constant $k$. So the masses would be at rest if they were at positions $\ell$ and $2\ell$.




Let the positions of the masses at time $t$ be $\ell + x_1(t)$ and $2\ell + x_2(t)$. (So the $x$'s are the displacement from the rest positions.) Then the masses obey the differential equations below.

$$\begin{aligned} mx_1''(t) &= -kx_1(t) \quad +k(x_2(t) - x_1(t)) \\ mx_2''(t) &= \qquad\qquad -k(x_2(t) - x_1(t)). \end{aligned}$$

(1) Find all solutions to these equations of the form $x_1(t) = a_1 \cos(\alpha t)$, $x_2(t) = a_2 \cos(\alpha t)$. (Hint: What does this have to do with eigenvalues?)
(2) Find a solution to these equations of the form $x_1 = a_1 \cos(\alpha t) + b_1 \cos(\beta t)$, $x_2 = a_2 \cos(\alpha t) + b_2 \cos(\beta t)$ with $x_1(0) = 0.1$ and $x_2 = -0.2$. (In other words, the masses start at positions $\ell + 0.1$ and $2\ell - 0.2$.