

PROBLEM SET ONE: DUE THURSDAY, JANUARY 13 AT 11:59 PM

See course website for homework policies.

Problem 1. We list some basic nutritional information:

- One cup of flour contains 80 grams of carbohydrate, 16 grams of protein and no fat.
- One egg contains no carbohydrate, 6 grams of protein and 5 grams of fat.
- One tablespoon of butter contains no carbohydrate or protein and 11 grams of fat.
- One cup of sugar contains 192 grams of carbohydrate and no protein or fat.

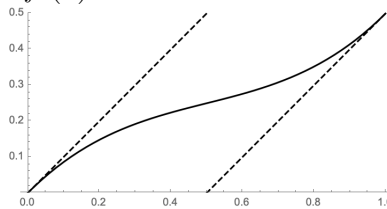
Compute the matrix R such that:

$$R \begin{bmatrix} \text{Flour} \\ \text{Eggs} \\ \text{Butter} \\ \text{Sugar} \end{bmatrix} = \begin{bmatrix} \text{Carbohydrate} \\ \text{Protein} \\ \text{Fat} \end{bmatrix}.$$

Solution: The total carbohydrates is $80(\text{flour}) + 0(\text{egg}) + 0(\text{butter}) + 192(\text{sugar})$. So the first row of the matrix is $[80 \ 0 \ 0 \ 192]$. Continuing in this way, the matrix is

$$\begin{bmatrix} 80 & 0 & 0 & 192 \\ 16 & 6 & 0 & 0 \\ 0 & 5 & 11 & 0 \end{bmatrix}.$$

Problem 2. The picture below depicts a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ with $f(0) = 0$, $f'(0) = 2$, $f(1) = 1$ and $f'(1) = 2$. Find the coefficients of $f(x)$.



Solution The equations $f(0) = 0$, $f'(0) = 2$, $f(1) = 1$ and $f'(1) = 2$ gives the linear equations:

$$\begin{aligned} d &= 0 \\ c &= 2 \\ a + b + c + d &= 1 \\ 3a + 2b + c &= 2 \end{aligned}$$

We perform row reduction. The job is easier if we make our pivots from right to left instead of left to right, so that the first two equations can be pivot rows with d and c . We box each variable as it becomes a pivot:

$$\begin{aligned} & & & & \boxed{d} &= 0 \\ & & & c & &= 2 \\ a + b + c & & & & &= 1 \\ 3a + 2b + c & & & & &= 2 \\ & & & & \boxed{d} &= 0 \\ & & & & \boxed{c} &= 2 \\ a + b & & & & &= -1 \\ 3a + 2b & & & & &= 0 \end{aligned}$$

$$\begin{array}{rcccc}
& & & \boxed{d} & = & 0 \\
& & & \boxed{c} & = & 2 \\
a & + & \boxed{b} & & = & -1 \\
a & & & & = & 2 \\
& & & & \boxed{d} & = & 0 \\
& & & \boxed{c} & = & 2 \\
& & \boxed{b} & & = & -3 \\
\boxed{a} & & & & = & 2
\end{array}$$

So the polynomial is $2x^3 - 3x^2 + 2x$.

Problem 3. Let A be an $\ell \times m$ matrix and let B be an $m \times n$ matrix.

- (1) Prove or disprove: If $\text{Ker}(A) = \{\vec{0}\}$ and $\text{Ker}(B) = \{\vec{0}\}$ then $\text{Ker}(AB) = \{\vec{0}\}$.
- (2) Prove or disprove: If $\text{Ker}(AB) = \{\vec{0}\}$ then $\text{Ker}(A) = \{\vec{0}\}$ and $\text{Ker}(B) = \{\vec{0}\}$.
- (3) Prove or disprove: If $\text{Im}(A) = \mathbb{R}^\ell$ and $\text{Im}(B) = \mathbb{R}^m$ then $\text{Im}(AB) = \mathbb{R}^\ell$.
- (4) Prove or disprove: If $\text{Im}(AB) = \mathbb{R}^\ell$ then $\text{Im}(A) = \mathbb{R}^\ell$ and $\text{Im}(B) = \mathbb{R}^m$.

Solution

Part 1: This is true. Let \vec{v} be in $\text{Ker}(AB)$, so $AB\vec{v} = \vec{0}$. Since $\text{Ker}(A) = \{\vec{0}\}$, we see that $B\vec{v} = \vec{0}$. Since $\text{Ker}(B) = \{\vec{0}\}$, we further deduce that $\vec{v} = \vec{0}$. So we have shown that $\vec{0}$ is the only element of $\text{Ker}(AB)$, as desired.

Part 2: This is false. A counterexample is $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 \end{bmatrix}$. The kernel of AB is $\{\vec{0}\}$, but $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in the kernel of A .

Part 3: This is true. Let \vec{z} be any vector in \mathbb{R}^ℓ . Since $\text{Im}(A) = \mathbb{R}^\ell$, we can find a vector \vec{y} in \mathbb{R}^m with $A\vec{y} = \vec{z}$. Since $\text{Im}(B) = \mathbb{R}^m$, we can find a vector \vec{x} in \mathbb{R}^n with $B\vec{x} = \vec{y}$. Then $AB\vec{x} = A\vec{y} = \vec{z}$, so we have shown that \vec{z} is in $\text{Im}(AB)$, as desired.

Part 4: This is false. Again, consider $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\text{Im}(AB)$ is all of \mathbb{R} , but $\text{Im}(B)$ is only a one dimensional subspace of \mathbb{R}^2 .

Problem 4. Let A be an $m \times n$ matrix and let V be an $n \times n$ invertible matrix.

- (1) Prove that $\text{Im}(AV) = \text{Im}(A)$.
- (2) Prove that $\text{Ker}(AV) = V^{-1}\text{Ker}(A)$. To spell this out more explicitly, show that $\text{Ker}(AV)$ is $\{V^{-1}\vec{x} : \vec{x} \text{ in } \text{Ker}(A)\}$.

Solution

Part 1: Let \vec{y} be a vector in \mathbb{R}^m . We must show that \vec{y} is in $\text{Im}(AV)$ if and only if \vec{y} is in $\text{Im}(A)$. First, suppose that \vec{y} is in $\text{Im}(AV)$, so $\vec{y} = AV\vec{x}$ for some \vec{x} . Then $\vec{y} = A(V\vec{x})$, so \vec{y} is in $\text{Im}(A)$. Conversely, suppose that \vec{y} is in $\text{Im}(A)$, so $\vec{y} = A\vec{x}$ for some \vec{x} . Then $\vec{y} = (AV)(V^{-1}\vec{x})$, so \vec{y} is in $\text{Im}(AV)$.

Part 2: Let \vec{x} be a vector in \mathbb{R}^n . We must show that $AV\vec{x} = \vec{0}$ if and only if $\vec{x} = V^{-1}\vec{w}$ with $A\vec{w} = \vec{0}$. The equation $\vec{x} = V^{-1}\vec{w}$ is the same as $V\vec{x} = \vec{w}$ and, indeed, $(AV)\vec{x} = A(V\vec{x})$, so $(AV)\vec{x} = \vec{0}$ if and only if $A(V\vec{x}) = \vec{0}$.

Problem 5. (Challenge) We described three row operations that we can perform on a matrix:

- (R1) Adding a multiple of row i to row j .
- (R2) Switching rows i and j .

(R3) Multiplying row i by a nonzero scalar c .

This problem investigates what we can do using by just doing (R1) many times.

- (1) Let \vec{x} and \vec{y} be two rows of A . Show that, by using (R1) three times, we can change these rows into $-\vec{y}$ and \vec{x} .
- (2) Let \vec{x} and \vec{y} be two rows of A and let c be a nonzero scalar. Show that, by using (R1) four times, we can change these rows into $c\vec{x}$ and $c^{-1}\vec{y}$.

Solution

We write the effect of our row operations on the two rows in question, omitting the others.

$$\mathbf{Part\ 1:} \quad \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \vec{x} \\ \vec{x} + \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\vec{y} \\ \vec{x} + \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\vec{y} \\ \vec{x} \end{bmatrix}$$

$$\mathbf{Part\ 2:} \quad \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \vec{x} \\ (c-1)\vec{x} + \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} c\vec{x} + \vec{y} \\ (c-1)\vec{x} + \vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} c\vec{x} + \vec{y} \\ c^{-1}\vec{y} \end{bmatrix} \rightsquigarrow \begin{bmatrix} c\vec{x} \\ c^{-1}\vec{y} \end{bmatrix}$$

In the third step of Part 2, we have subtracted $1 - c^{-1}$ times the first row from the second one.