Solution Set Ten

8.2.1 This is the set of vectors $\left[\begin{smallmatrix} w & x & y & z \end{smallmatrix}\right]$ with $w - y + z = 0$ and $2w + 3x - y + 2z = 0$. Row reducing, we find that a basis for the solutions to these equations is $\left[\begin{smallmatrix} -1 & 0 & 0 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 3 & -1 & 3 & 0 \end{smallmatrix}\right]$.

8.2.2 We first make the vectors orthogonal. β_1 and β_2 are already orthogonal. The projection of β_3 onto $\mathrm{Span}(\beta_1, \beta_2)$ is $\frac{\langle \beta_1, \beta_3 \rangle}{\langle \beta_1, \beta_1 \rangle}\beta_1 + \frac{\langle \beta_2, \beta_3 \rangle}{\langle \beta_2, \beta_2 \rangle}\beta_2 = \frac{4}{2}\left[\begin{smallmatrix} 1 & 0 & 1 \end{smallmatrix}\right] + \frac{-4}{2}\left[\begin{smallmatrix} 1 & 0 & -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 & 4 \end{smallmatrix}\right]$. So the component of β_3 orthogonal to $\mathrm{Span}(\beta_1, \beta_2)$ is $\beta_3 - \left[\begin{smallmatrix} 0 & 0 & 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 3 & 0 \end{smallmatrix}\right]$.

So we now have orthogonal vectors $\left[\begin{smallmatrix} 1 & 0 & 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 & -1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 3 & 0 \end{smallmatrix}\right]$. We rescale these to be orthonormal, giving:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

8.2.12 Write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in W$ and $\alpha_2 \in W^\perp$, and similarly write $\beta = \beta_1 + \beta_2$. Then $E(\alpha) = \alpha_1$ and $E(\beta) = \beta_1$. We have $\langle E(\alpha), \beta \rangle = \langle \alpha_1, \beta_1 + \beta_2 \rangle = \langle \alpha_1, \beta_1 \rangle + \langle \alpha_1, \beta_2 \rangle = \langle \alpha_1, \beta_1 \rangle$ since $\alpha_1 \in W$ and $\beta_2 \in W^\perp$. Similarly, $\langle \alpha, E(\beta) \rangle = \langle \alpha_1 + \alpha_2, \beta_1 \rangle = \langle \alpha_2, \beta_1 \rangle + \langle \alpha_1, \beta_1 \rangle = \langle \alpha_1, \beta_1 \rangle$. So both expressions are equal to $\langle \alpha_1, \beta_1 \rangle$.

8.4.4 Let the columns of U be $\vec{u}$, $\vec{v}$. The vector $\vec{u}$ must have length 1, so we can write it as $\left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$. The vector $\vec{v}$ must be orthogonal to this, so it is a scalar multiple of $\left[\begin{smallmatrix} \sin\theta \\ -\cos\theta \end{smallmatrix}\right]$ and, since $\vec{v}$ is length 1, we have $\vec{v} = \pm\left[\begin{smallmatrix} \sin\theta \\ -\cos\theta \end{smallmatrix}\right]$.

We now answer the various questions:

$$U_\theta U_\phi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} =$$
$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = U_{\theta+\phi}.$$

This makes sense: Rotation by θ followed by rotation by ϕ is rotation by $\theta + \phi$.

$$U_\theta^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^* = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = U_{-\theta}.$$

Note that, for a real matrix, we have $A^* = A^T$.

The matrix of U_θ in the rotated basis is $U_\phi U_\theta U_\phi^{-1} = U_\theta$, since θ and ϕ commute.

8.4.8 We have

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} = \begin{bmatrix} \frac{e^{i\theta}+e^{-i\theta}}{2} & \frac{-e^{i\theta}+e^{-i\theta}}{2i} \\ \frac{e^{i\theta}-e^{-i\theta}}{2i} & \frac{e^{i\theta}+e^{-i\theta}}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Problem 1. In this problem, we will prove the following result: Let A be a square matrix and suppose that the characteristic polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) = \prod(x-\lambda_i)^{n_i}$. Then there is a basis in which A is upper triangular.

(1) Let V be an m-dimensional vector space and let $C : V \to V$ be a linear transformation with $C^m = 0$. Show that V has a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ such that $C(\vec{v}_i) \in \mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1})$. Conclude that, in this basis, C is upper triangular with 0's on the diagonal.

(2) Let V be an m-dimensional vector space, let λ be a scalar and let $B : V \to V$ be a linear transformation with $\chi_B(x) = (x-\lambda)^m$. Show that there is a basis for V in which B is upper triangular with λ's on the diagonal.

(3) Let V be an m-dimensional vector space, let $A : V \to V$ be a linear transformation and suppose that the minimal polynomial $\chi_A(x)$ factors into linear factors $\chi_A(x) =$

$\prod(x-\lambda_i)^{n_i}$. Show that there is a basis for V where A is upper triangular with the λ_i on the diagonal.

Solution (1): We show, by induction on j, that we can find linearly independent vectors $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_j$ such that, for $i \leq j$, we have $C\vec{v}_i \in \mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1})$. The base case, $j=0$, is clear.

So, suppose that we have constructed $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_{j-1}$ as above, and put $W = \mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1})$. We need to show that, if W is not equal to all of V, we can find some $\vec{v}_j \notin W$ such that $C\vec{v}_j \in W$. Take any $\vec{v}$ not in W and compute $\vec{v}$, $C\vec{v}$, $C^2\vec{v}$, etcetera. Since $C^m = 0$, we eventually have $C^k\vec{v} \in W$; let k be the index such that $C^{k-1}\vec{v} \notin W$ and $C^k\vec{v} \in W$. Take $\vec{v}_j = C^{k-1}\vec{v}$.

Solution (2): Apply part (1) to $C := B - \lambda\,\mathrm{Id}$.

Solution (3): By the primary decomposition theorem, we can choose a basis where A becomes block diagonal as $\begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix}$ where B_i has characteristic polynomial $(x-\lambda_i)^{n_i}$. Then, by the previous part, we can make each B_i upper triangular with diagonal entries λ_i.

Problem 2. Let F be a field and let $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ be an ***irreducible*** polynomial with coefficients in F.

(1) Let V be an n-dimensional vector space and let $A : V \to V$ be a linear transformation with $\chi_A(x) = f(x)$. Let $\vec{v}$ be any nonzero vector in V. Show that $\vec{v}$, $A\vec{v}$, $\ldots$, $A^{n-1}\vec{v}$ is a basis of V.
(2) Let A and V be as in the previous part. Write the matrix of A in the basis $\vec{v}$, $A\vec{v}$, $\ldots$, $A^{n-1}\vec{v}$.

Solution (1): There is more than one way to do this, here is the shortest I found. Suppose, to the contrary, that there is some $k < n$ with $A^k\vec{v} \in \mathrm{Span}(\vec{v}, A\vec{v}, \ldots, A^{k-1}\vec{v})$, and choose the minimal such k. Put $W = \mathrm{Span}(\vec{v}, A\vec{v}, \ldots, A^{k-1}\vec{v})$, then A maps W to itself, and $\dim W = k < n$. So the characteristic polynomial of $A|_W$ divides $\chi_A(x)$. But this contradicts that $\chi_A(x)$ is irreducible.

Solution (2): Now that we know that $\vec{v}$, $A\vec{v}$, $\ldots$, $A^{n-1}\vec{v}$ is a basis, this question makes sense. For $0 \leq k < n-1$, we have $A(A^k\vec{v}) = A^{k+1}\vec{v}$, so the first $n-1$ columns of A have a 1 in position $(k+1, k)$ and 0's elsewhere. For the last column, we compute

$$A(A^{n-1}\vec{v}) = A^n\vec{v} = -\left(f_{n-1}A^{n-1} + \cdots + f_1 A + f_0\right)\vec{v} = -f_{n-1}A^{n-1}(\vec{v}) - \cdots - f_1 A(\vec{v}) - f_0\vec{v}$$

where the middle equality is the Cayley-Hamilton theorem. We conclude that the matrix of A in this basis is

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & -f_0 \\ 1 & 0 & 0 & \cdots & \cdots & -f_1 \\ 0 & 1 & 0 & \cdots & \cdots & -f_2 \\ 0 & 0 & 1 & \ddots & & -f_3 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -f_{n-1} \end{bmatrix}.$$

Problem 3. Let V be the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V by

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

(1) Show that the following list of functions is orthonormal: $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}}\sin(nx)$ for $n \geq 1$, and $\frac{1}{\sqrt{\pi}}\cos(nx)$ for $n \geq 1$.

(2) Let $f(x) = x$. Find the function in $\operatorname{Span}(\sin x, \sin(2x), \sin(3x))$ which is closest to the function $f(x)$.

Solution (1): We first check that each of these functions has length 1:

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{dx}{\sqrt{2\pi}^2} = \frac{2\pi}{2\pi} = 1.$$

$$\left\langle \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\sin^2(nx)dx}{\sqrt{\pi}^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(nx)dx =$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1-\cos(2nx)}{2}\right)dx = \frac{1}{\pi}\left(\frac{2\pi}{2} - 0\right) = 1.$$

$$\left\langle \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} \frac{\cos^2(nx)dx}{\sqrt{\pi}^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(nx)dx =$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos(2nx)}{2}\right)dx = \frac{1}{\pi}\left(\frac{2\pi}{2} + 0\right) = 1.$$

Next, we check the orthogonality claim. At this point, we can drop out the constants $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{\pi}}$, since we are just trying to prove that things are 0.

$$\langle 1, \sin(nx)\rangle = \int_{-\pi}^{\pi} \sin(nx)dx = 0.$$

$$\langle 1, \cos(nx)\rangle = \int_{-\pi}^{\pi} \cos(nx)dx = 0.$$

$$\langle \cos(mx), \sin(nx)\rangle = \int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\sin((m+n)x) - \sin((m-n)x)\right)dx = 0.$$

And, for $m \neq n$:

$$\langle \cos(mx), \cos(nx)\rangle = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos((m+n)x) + \cos((m-n)x)\right)dx = 0.$$

$$\langle \sin(mx), \sin(nx)\rangle = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos((m+n)x) - \cos((m-n)x)\right)dx = 0.$$

Solution (2): We need to orthogonally project x onto $\operatorname{Span}(\sin(x), \sin(2x), \sin(3x))$. So the coefficient of $\sin(nx)$ is

$$\frac{\langle x, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle}.$$

We compute

$$\langle x, \sin(nx)\rangle = \int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2\pi(-1)^{n+1}}{n} \quad \text{and} \quad \langle \sin(nx), \sin(nx)\rangle = \int_{-\pi}^{\pi} \sin^2(nx)dx = \pi.$$

So the orthogonal projection of x onto $\operatorname{Span}(\sin(x), \sin(2x), \sin(3x))$ is

$$2\sin(x) - \tfrac{2}{2}\sin(2x) + \tfrac{2}{3}\sin(3x).$$