Solution Set Eleven

8.5.1 We go through four steps: Compute the characteristic polynomial, compute the eigenvalues, compute the eigenvectors, make them orthogonal:

matrix	$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$	$\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$
char. poly.	x^2-2x	x^2-4x+3	x^2-1
eigenvalues	$0,\ 2$	$1,\ 3$	$1,\ -1$
eigenvectors	$\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right],\ \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}\sin\theta\\ 1-\cos\theta\end{smallmatrix}\right],\ \left[\begin{smallmatrix}-\sin\theta\\ 1+\cos\theta\end{smallmatrix}\right]$
matrices	$P=\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$ $A=\left[\begin{smallmatrix}0 & 0\\ 0 & 2\end{smallmatrix}\right]$	$P=\frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right]$ $A=\left[\begin{smallmatrix}1 & 0\\ 0 & 3\end{smallmatrix}\right]$	$P=\left[\begin{smallmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{smallmatrix}\right]$ $A=\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$

The computation with P in the bottom right is not obvious, so we explain: We have $(\sin\theta)^2 + (1-\cos\theta)^2 = \cos^2\theta + \sin^2\theta - 2\cos\theta + 1 = 2 - 2\cos\theta = 4\sin^2(\theta/2)$, so the first column of P is $\frac{1}{2\sin\theta(\theta/2)}\left[\begin{smallmatrix}\sin\theta\\ 1-\cos\theta\end{smallmatrix}\right] = \left[\begin{smallmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{smallmatrix}\right]$. Similarly in the second column, we have $(-\sin\theta)^2 + (1+\cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\cos\theta + 1 = 2\cos\theta + 2 = 4\cos^2(\theta/2)$ so the second column of P is $\frac{1}{2\cos(\theta/2)}\left[\begin{smallmatrix}-\sin\theta\\ 1+\cos\theta\end{smallmatrix}\right] = \left[\begin{smallmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{smallmatrix}\right]$.

8.5.3 This time, we just need to find the eigenvalues. The characteristic polynomial of A is $x^3 - 9x^2 - 6 = x(x^2 - 9x - 6)$, so the eigenvalues are 0 and $\frac{9\pm\sqrt{105}}{2}$. So $D = \begin{bmatrix}0 & & \\ & \frac{9+\sqrt{105}}{2} & \\ & & \frac{9-\sqrt{105}}{2}\end{bmatrix}$.

8.5.6 Let $T = U^\dagger D U$ for a unitary matrix U.

The operator T is self-adjoint (also called Hermitian) if and only if $T = T^\dagger$, or, in other words $U^\dagger D U = (U^\dagger D U)^\dagger = U^\dagger D^\dagger U^{\dagger\dagger}$. We have $D^\dagger = \overline{D}$, since D is diagonal, and $U^{\dagger\dagger} = U$, so this simplifies to $U^\dagger D U = U^\dagger \overline{D} U$ or $D = \overline{D}$. This happens if and only if each eigenvalue λ of D obeys $\lambda = \overline{\lambda}$, so λ is real.

The operator T is positive (also called positive definite) if and only if $T = T^\dagger$ and, for all $\vec{x} \neq \vec{0}$, we have $\vec{x}^\dagger T \vec{x}$. As we checked above, this means that the eigenvalues are real, so T is positive definite if and only if $\vec{x}^\dagger U^\dagger D U \vec{x} > 0$ for all $\vec{x} \neq 0$. We can regroup this equation as $(U\vec{x})^\dagger D (U\vec{x})$ and, since U is invertible, the vector $U\vec{x}$ ranges over all vectors. So we are requiring that $\vec{y}^\dagger D \vec{y} > 0$ for all nonzero $\vec{y}$. We expand $\vec{y}^\dagger D \vec{y} = \sum_{i=1}^n \overline{y}_i \lambda_i y_i = \sum \lambda_i |y_i|^2$ where the λ_i are the eigenvalues of D and the y_i are the entries of $\vec{y}$. It is clear that requiring that $\sum \lambda_i |y_i|^2 > 0$ for all nonzero $\vec{y}$ is the same as imposing that all the λ_i are > 0.

Finally, T is unitary if and only if $T^\dagger T = \mathrm{Id}$. We compute $T^\dagger T = (U^\dagger D U)^\dagger (U^\dagger D U) = U^\dagger D^\dagger U^{\dagger\dagger} U^\dagger D U = U^\dagger D^\dagger D U$ since U is unitary. So $U^\dagger D^\dagger D U = \mathrm{Id}$ if and only if $D^\dagger D = \mathrm{Id}$. Since D is diagonal, with diagonal entries λ_i, the matrix $D^\dagger D$ has diagonal entries $\overline{\lambda_i}\lambda_i = |\lambda_i|^2$. So $D^\dagger D = \mathrm{Id}$ if and only if $|\lambda_i|^2 = 1$ for each λ_i, as required.

8.5.9 Since A is a real symmetric matrix, we can write $A = Q^T D Q$ for some orthogonal matrix Q and some diagonal matrix D. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the diagonal entries of D. Then $A = B^3$

where

$$B = Q^T \begin{bmatrix} \sqrt[3]{\lambda_1} & & & \\ & \sqrt[3]{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt[3]{\lambda_n} \end{bmatrix} Q.$$

8.5.10 Since A is positive, we can write $A = Q^T DQ$ where Q is orthogonal and D is diagonal with entries that are positive real numbers. Then $A = B^2$ where

$$B = Q^T \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix} Q.$$

8.5.11 For finite dimensional vector spaces, one can prove this using our results about diagonalizability. But it is nicer to prove it without them, with the added benefit of handling the infinite dimensional case.

So, let A be normal and suppose that $A^n = 0$. We must show that A is 0. We will show that having $A^n = 0$ implies that $A^{n-1} = 0$; continuing in this manner, we will deduce that $A = 0$.

We have $n \leq 2n-2$, so $(A^\dagger)^{2n-2}A^{2n-2} = 0$. Using the normality, this shows that $(A^\dagger)^{n-1}A^{n-1}(A^\dagger)^{n-1}A^{n-1}$. So, for any vector $\vec{v}$, we have $\langle (A^\dagger)^{n-1}A^{n-1}(A^\dagger)^{n-1}A^{n-1}\vec{v}, \vec{v}\rangle = 0$. We rearrange this to $\langle (A^\dagger)^{n-1}A^{n-1}\vec{v}, (A^\dagger)^{n-1}A^{n-1}\vec{v}\rangle = 0$, so $(A^\dagger)^{n-1}A^{n-1}\vec{v} = \vec{0}$. But then $\langle (A^\dagger)^{n-1}A^{n-1}\vec{v}, \vec{v}\rangle = 0$, which we can rearrange to $\langle A^{n-1}\vec{v}, A^{n-1}\vec{v}\rangle$. We deduce that $A^{n-1}\vec{v} = \vec{0}$ for any $\vec{v}$, and thus $A^{n-1} = 0$. Continuing in this manner, we will deduce that $A = 0$.

Problem 1. Let V be the vector space of smooth (meaning infinitely differentiable) functions $[0, 2\pi] \to \mathbb{R}$ which obey $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$. Define an inner product on V by

$$\langle f(x), g(x)\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Define the linear operator $L : V \to V$ by $L(f) = \frac{d^2}{(dx)^2}f$. Show that L is selfadjoint, meaning that $\langle L(f), g\rangle = \langle f, L(g)\rangle$.

Solution We have $\langle L(f), g\rangle = \int_0^{2\pi} \frac{d^2 f}{(dx)^2}(x)g(x)dx$ and $\langle f, L(g)\rangle = \int_0^{2\pi} f(x)\frac{d^2 g}{(dx)^2}(x)dx$. We now integrate by parts:

$$\int_0^{2\pi} \frac{d^2 f}{(dx)^2}(x)g(x)dx = f'(x)g(x)|_0^{2\pi} - \int_0^{2\pi} f'(x)g'(x)dx = -\int_0^{2\pi} f'(x)g'(x)dx.$$

In the second equality, we have used that $f'(0) = f'(2\pi)$ and $g(0) = g(2\pi)$. But, similarly,

$$\int_0^{2\pi} f(x)\frac{d^2 g}{(dx)^2}(x)dx = f(x)g'(x)|_0^{2\pi} - \int_0^{2\pi} f'(x)g'(x)dx = -\int_0^{2\pi} f'(x)g'(x)dx.$$

So $\langle L(f), g\rangle = \langle f, L(g)\rangle$ as desired.

Problem 2. Let A be a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. In this problem, we will show that A has a singular value decomposition, meaning that we can find two orthonormal bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ for $\mathbb{R}^n$ such that $A\vec{u}_i$ is a scalar multiple of $\vec{v}_i$ for each $1 \leq i \leq n$.

(1) Consider the function $|A\vec{x}|$ on the unit sphere $\{\vec{x}|\langle\vec{x},\vec{x}\rangle = 1\}$. Let $\vec{u}$ be the vector on the unit sphere where $|A\vec{u}|$ is maximized. Define $\vec{v} = A\vec{u}$. Show that A takes $\vec{u}^{\perp}$ to $\vec{v}^{\perp}$.

(2) Show (induct on n) there there is a pair of orthonormal bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ for $\mathbb{R}^n$ such that $A\vec{u}_i$ is a scalar multiple of $\vec{v}_i$ for each $1 \le i \le n$.

Solution (1): Let $\vec{x}$ be orthogonal to $\vec{u}$ and let $\vec{y} = A\vec{x}$. Normalize $\vec{x}$ to have length 1. We want to show that $\vec{y}$ is orthogonal to $\vec{v}$.

Consider $|A((\cos\theta)\vec{u} + (\sin\theta)\vec{x})|$. Since $(\cos\theta)\vec{u} + (\sin\theta)\vec{x}$ has length 1 for all θ, the function $|A((\cos\theta)\vec{u} + (\sin\theta)\vec{x})|$ should have a local minimum at $\theta = 0$. We compute

$$|A((\cos\theta)\vec{u} + (\sin\theta)\vec{x})|^2 = \langle A((\cos\theta)\vec{u} + (\sin\theta)\vec{x}),\ A((\cos\theta)\vec{u} + (\sin\theta)\vec{x})\rangle =$$
$$\cos^2\theta\langle A\vec{u}, A\vec{u}\rangle + \cos\theta\sin\theta\,(\langle A\vec{u}, A\vec{x}\rangle + \langle A\vec{x}, A\vec{u}\rangle) + \sin^2\theta\langle A\vec{x}, A\vec{x}\rangle.$$

Taking the derivative with respect to θ, we get that $\langle A\vec{u}, A\vec{x}\rangle + \langle A\vec{x}, A\vec{u}\rangle = 0$. Since $\langle\ ,\ \rangle$ is symmetric, we conclude that $\langle A\vec{u}, A\vec{x}\rangle = 0$.

Solution (2): We will show by induction on n that there are bases orthonormal bases $(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ such that $A\vec{u}_i$ is a scalar multiple of $\vec{v}_i$ for each $1 \le i \le n$. By the argument above, we can find a unit vector $\vec{u}$ such that A carries $\vec{u}^{\perp}$ to $(A\vec{u})^{\perp}$. Put $\sigma = |A\vec{u}|$. If $\sigma \neq 0$, put $\vec{v} = (A\vec{u})/\sigma$; if $\sigma = 0$, then take $\vec{v}$ to be any unit vector orthogonal to $(A\vec{u})^{\perp}$. In either case, we have $\sigma\vec{v} = A\vec{u}$ and A carries $(\vec{u})^{\perp}$ to $(\vec{v})^{\perp}$. Inductively, we can find orthonormal bases $\vec{u}_2$, $\vec{u}_3$, $\ldots$, $\vec{u}_n$ for $(\vec{u})^{\perp}$ and $\vec{v}_2$, $\vec{v}_3$, $\ldots$, $\vec{v}_n$ for $(\vec{v})^{\perp}$ such that $A\vec{u}_i = \sigma_i\vec{v}_i$ for some σ_i. Then $(\vec{u}, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_n)$ and $(\vec{v}, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)$ are the orthonormal bases we seek.