Problem 1. Find linear polynomials $at+b$ and $ct+d$ such that

$$(at+b)(t^2+1)+(ct+d)(t^2+t+1)=1.$$

Solution: Expanding the products, we get

$$at^3+bt^2+at+b+ct^3+(c+d)t^2+(c+d)t+d=1.$$

Equating the coefficients of t^3, t^2, t and 1, we get

$$\begin{array}{cccccc} a & & +c & & = & 0 \\ & b & +c & +d & = & 0 \\ a & & +c & +d & = & 0 \\ & b & & +d & = & 1 \end{array}.$$

We now run the row reduction algorithm. I'll put pivots in boxes as I create them.

$$\begin{array}{cccccc} a & & +c & & = & 0 \\ & b & +c & +d & = & 0 \\ a & & +c & +d & = & 0 \\ & b & & +d & = & 1 \end{array} \rightsquigarrow \begin{array}{cccccc} \boxed{a} & & +c & & = & 0 \\ & b & +c & +d & = & 0 \\ & & & d & = & 0 \\ & b & & +d & = & 1 \end{array} \rightsquigarrow \begin{array}{cccccc} \boxed{a} & & +c & & = & 0 \\ & \boxed{b} & +c & +d & = & 0 \\ & & & d & = & 0 \\ & & -c & +d & = & 1 \end{array} \rightsquigarrow$$

$$\rightsquigarrow \begin{array}{cccccc} \boxed{a} & & & +d & = & 1 \\ & \boxed{b} & & +2d & = & 1 \\ & & \boxed{c} & -d & = & -1 \\ & & & d & = & 0 \end{array} \rightsquigarrow \begin{array}{cccccc} \boxed{a} & & & & = & 1 \\ & \boxed{b} & & & = & 1 \\ & & \boxed{c} & & = & -1 \\ & & & \boxed{d} & = & 0 \end{array}.$$

So $(a,b,c,d)=(1,1,-1,0)$ and the solution is $(t+1)(t^2+1)-t(t^2+t+1)=1$.

Problem 2. Find a nonzero vector which is both in Image $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and in Image $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$.

We want to solve

$$p\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}+q\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}=r\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+s\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

or, in other words,

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We run the row reduction algorithm:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & -3 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

So we deduce that $p-2s=q+s=r-s=0$. So a solution to the linear equations is $(p,q,r,s)=(2,-1,1,1)$. Returning to the original question, we see that

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}-\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

So $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (or any scalar multiple of it) is in the image of both matrices.

Problem 3. Let F be a field (see Section 1.1) in your textbook. Prove the following from the axioms of a field; you may also use the results that $0x=x0=0$ and $(-1)x=x(-1)=-x$.

(1) For any x in F, we have $x^2-1=(x+1)(x-1)$.
(2) For any elements x and y in F, if $xy=0$ then either $x=0$ or $y=0$.
(3) For any x in F, if $x^2=1$ then $x=1$ or $x=-1$.

Solution:

Part (1): Using the distributive law repeatedly: $(x+1)(x-1) = x(x-1) + 1(x-1) = (x^2 + x(-1)) + (x-1)$. Using the associative law twice, this is $((x^2 + x(-1)) + x) - 1 = (x^2 + (x(-1) + x)) - 1$. Using $x(-1) = -x$, this is $(x^2 + (-x + x)) - 1 = (x^2 + 0) - 1 = x^2 - 1$.

Part (2): We break into two cases: $x = 0$ or $x \neq 0$. If $x = 0$, we are done. If not, multiply both sides of the equation by x^{-1} to give $x^{-1}(xy) = x^{-1}0 = 0$. We then rewrite the left hand side using the associative rule: $x^{-1}(xy) = (x^{-1}x)y = 1 \cdot y = y$. So we have shown that $y = 0$.

Part (3): We use the two previous parts! If $x^2 = 1$ then $x^2 - 1 = 0$, and we showed that $x^2 - 1 = (x+1)(x-1)$. So, by the second part, either $x - 1 = 0$ or $x + 1 = 0$. Adding 1 or -1 respectively to both sides, we get $x = 1$ or $x = -1$.

Problem 4. Let A be an $\ell \times m$ matrix and let B be an $m \times n$ matrix.

(1) Suppose that $\mathrm{Ker}(AB) = 0$ and $\mathrm{Image}(B) = \mathbb{R}^m$. Show that $\mathrm{Ker}(A) = \{\vec{0}\}$.
(2) Suppose that $\mathrm{Image}(AB) = \mathbb{R}^\ell$ and $\mathrm{Ker}(A) = \{\vec{0}\}$. Show that $\mathrm{Image}(B) = \mathbb{R}^m$.

Solution:

Part 1: Suppose that $A\vec{y} = \vec{0}$; we must show that $\vec{y} = 0$. Since B is surjective, we can find some $\vec{x}$ with $B\vec{x} = \vec{y}$, so $AB\vec{x} = \vec{0}$. By our assumption that $\mathrm{Ker}(AB) = 0$, this means that $\vec{x} = \vec{0}$. We deduce that $\vec{y} = A\vec{x} = A\vec{0} = \vec{0}$.

Part 2: Let $\vec{y} \in \mathbb{R}^m$, we must find $\vec{x}$ with $B\vec{x} = \vec{y}$. By the hypothesis on AB, we can find a vector $\vec{x}$ with $(AB)\vec{x} = A\vec{y}$. Then, by the injectivity of A, we have $B\vec{x} = \vec{y}$.

Problem 5. Let X be a set and let F be a field. Let F^X be the vector space of all functions $f : X \to F$. (See Example 3 in Section 2.1 of your textbook.) Let F^X_{finite} be the set of functions $f : X \to F$ such that $\{x \in X : f(x) \neq 0\}$ is finite. Show that F^X_{finite} is a subspace of F^X.

Solution: We must show that F^X_{finite} is closed under addition and scalar multiplication. For addition, suppose that f and g are functions in F^X_{finite}. Then $\{x : f(x) + g(x) \neq 0\}$ is a subset of $\{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\}$, The case of scalar multiplication is even easier: Let $f \in F^X_{\text{finite}}$ and let a be a scalar. If $a \neq 0$, then $\{x : af(x) \neq 0\} = \{x : f(x) \neq 0\}$, so it is finite. If $a = 0$ then $af(x)$ is 0 everywhere. So, either way, $af(x)$ is in F^X_{finite}.

Problem 6. Let T be the set of functions $\mathbb{R} \to \mathbb{R}$ which are of the form $a \cos t + b \sin t$. For $x + iy$ in $\mathbb{C}$ and $f(t)$ in T, define $(x + iy) * f(t) = xf(t) + y\frac{df}{dt}$. Show that T is a $\mathbb{C}$-vector space with respect to this scalar multiplication, and the usual addition.

Solution: Since the addition operation is ordinary addition, it is clearly commutative, associative, has a $\vec{0}$ and has additive inverses. We now check the conditions involving multiplication.

We have $(1 + 0i)f(t) = f(t)$, checking the axiom of the multiplicative identity.

To check distributivity with two scalars and one vector, we must check that

$$((x_1 + x_2) + i(y_1 + y_2))f(t) = (x_1 + iy_1)f(t) + (x_2 + iy_2)f(t).$$

Indeed, this expands to

$$(x_1 + x_2)f(t) + (y_1 + y_2)\frac{df}{ft} = x_1 f(t) + y_1 \frac{df}{ft} + x_2 f(t) + y_2 \frac{df}{dt}.$$

To check distributivity with one scalar and two vectors, we must check that

$$(x + iy)(f + g) = (x + iy)f + (x + iy)g.$$

Indeed, this expands to

$$xf + xg + y\tfrac{df}{dt} + y\tfrac{dg}{dt}.$$

The most interesting, and hardest thing, to check is associativity. We want to show that

$$(x_1 + iy_1)\big((x_2 + iy_2)f(t)\big) = \big((x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\big)f(t).$$

The left hand side is

$$(x_1 + iy_1)(x_2 f + y_2\tfrac{df}{dt}) = x_1x_2 f + x_1y_2\tfrac{df}{dt} + x_2y_1\tfrac{df}{dt} + y_1y_2\tfrac{d^2f}{(dt)^2}.$$

The right hand side is

$$(x_1x_2 - y_1y_2)f + (x_1y_2 + x_2y_1)\tfrac{df}{dt}.$$

But now notice that, for any $f(t)$ in T, we have $\frac{d^2f}{(dt)^2} = -f$! So the two formulas match.