

2.2.2

- (a) This is not a subspace. For example, the function $f(x) = 1$ obeys this condition, but the function $f(x) = 2$ does not, so this set is not closed under multiplication by 2.
- (b) This is a subspace. To check closure under addition note that, if $f(0) = f(1)$ and $g(0) = g(1)$ then $f(0) + g(0) = f(1) + g(1)$; to check closure under scalar multiplication, note that, if $f(0) = f(1)$ and c is any scalar, then $cf(0) = cg(0)$.
- (c) This is not a subspace. One reason is that it does not contain 0.
- (d) This is a subspace. To check closure under addition note that, if $f(-1) = 0$ and $g(-1) = 0$, then $f(-1) + g(-1) = 0$; to check closure under scalar multiplication, note that, if $f(-1) = 0$ and c is any scalar, then $cf(-1) = 0$.
- (e) This is a subspace: The sum of two continuous functions is continuous, and the product of a continuous function with a scalar is continuous.

2.2.3 We need to check whether the equations

$$x \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 1 \\ -3 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 9 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

are solvable. In other words,

$$\begin{aligned} 2x - y + z &= 3 \\ -x + y + z &= -1 \\ 3x + y + 9z &= 0 \\ 2x - 3y - 5z &= -1 \end{aligned} .$$

Row reducing, we get

$$\begin{aligned} x + 2z &= 0 \\ y + 3z &= 0 \\ 0 &= 1 \\ 0 &= 0 \end{aligned} .$$

Since the equation $0 = 1$ has no solutions, these equations are not solvable. (There are many other ways to reach this conclusion.)

2.2.8

- (a) We first check closure under addition: If $f(x) = \pm f(-x)$ and $g(x) = \pm g(-x)$ (with the same sign) then $(f + g)(x) = \pm(f + g)(-x)$. Similarly, if $f(x) = \pm f(-x)$ and c is any scalar, then $cf(x) = \pm cf(-x)$.
- (b) For any function $f(x)$, we have $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$. The first function is in V_e and the second is in V_o .
- (c) Suppose that $f(x)$ is in V_e and in V_o . So, for every $x \in \mathbb{R}$, we have $f(x) = f(-x)$ and $f(x) = -f(-x)$. So $f(-x) = -f(-x)$, and we deduce that $f(-x) = 0$ for all x .

2.2.9 By assumption, the vectors α_1 and α_2 exist, so we just must prove uniqueness. Suppose that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ for $\alpha_1, \beta_1 \in W_1$ and $\alpha_2, \beta_2 \in W_2$. Then $\alpha_1 - \beta_1 = \beta_2 - \alpha_2$. Since W_1 and W_2 are closed under subtraction, we have $\alpha_1 - \beta_1 \in W_1$ and $\beta_2 - \alpha_2 \in W_2$, so $\alpha_1 - \beta_1 = \beta_2 - \alpha_2 \in W_1 \cap W_2 = \{0\}$. We deduce that $\alpha_1 - \beta_1 = \beta_2 - \alpha_2 = 0$, so $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. We have proved uniqueness. \square

2.3.1 Let \vec{u} and \vec{v} be linearly dependent. So there exist scalars a and b , not both 0, so that $a\vec{u} + b\vec{v} = \vec{0}$. At least one of a and b is not zero; without loss of generality, suppose that

$b \neq 0$. Then $\vec{v} = -\frac{a}{b}\vec{u}$, so \vec{v} is a scalar multiple of \vec{u} . (If a were not zero, then \vec{u} would be a scalar multiple of \vec{v} .)

2.3.2 We row reduce:

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since some of the rows are not pivot rows, the vectors are not linearly independent. (We could have made these into column vectors instead, in which case we would have looked to see if there were non-pivot columns.)

2.3.3 We can use the nonzero rows of the matrix we computed above: $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$.

2.3.5 A simple example is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Problem 1. Let U , V and W be vector spaces (over some field F). Let f_1, f_2 be linear transformations $U \rightarrow V$ and let g_1 and g_2 be linear transformations $V \rightarrow W$.

- (1) Show that $(g_1 + g_2)f_1 = g_1f_1 + g_2f_1$.
- (2) Show that $g_1(f_1 + f_2) = g_1f_1 + g_1f_2$.

As a reminder, $f_1 + f_2$ is the linear transformation defined by $(f_1 + f_2)(\vec{u}) = f_1(\vec{u}) + f_2(\vec{u})$ and the other sums are defined likewise; the product g_1f_1 is the linear transformation $U \rightarrow W$ defined by $(g_1f_1)(\vec{u}) = g_1(f_1(\vec{u}))$.

Solution We evaluate the left and right hand sides of each formula on a general element \vec{u} of U . For the first equation, we have

$$((g_1 + g_2)f_1)(\vec{u}) = (g_1 + g_2)(f_1(\vec{u})) = g_1(f_1(\vec{u})) + g_2(f_1(\vec{u})) = (g_1f_1)(\vec{u}) + (g_2f_1)(\vec{u}) = (g_1f_1 + g_2f_1)(\vec{u}).$$

Here we have just used the definitions of composition and addition. For the second, we have

$$(g_1(f_1 + f_2))(\vec{u}) = g_1((f_1 + f_2)(\vec{u})) = g_1(f_1(\vec{u}) + f_2(\vec{u})) = g_1(f_1(\vec{u})) + g_1(f_2(\vec{u})).$$

In the last step, we have used that g_1 is linear.

Problem 2. Let A be a 2×4 real matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ such that

- \vec{a}_1 is nonzero.
- $\vec{a}_2 = 3\vec{a}_1$.
- \vec{a}_3 is not in a multiple of \vec{a}_1 .
- $\vec{a}_4 = 5\vec{a}_1 + 7\vec{a}_3$.

Find the row reduction of A and prove your answer to be correct.

Solution The answer is

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}.$$

Since the first column is nonzero, it is a pivot column in the row reduction. Then the second column is 3 times it. Since the third column is not a multiple of the first column, it is also a pivot column. The last condition lets us compute the 4-th column.

Problem 3. Let $(x_1, y_1), (x_2, y_2), \dots, (x_6, y_6)$ be six points in the plane \mathbb{R}^2 . By a quadratic polynomial, we mean a polynomial of the form $f(x, y) = f_{00} + f_{10}x + f_{01}y + f_{20}x^2 + f_{11}xy + f_{02}y^2$. Show that **exactly one** of the following conditions holds. Hint: What does this have to do with linear algebra?

- There is a nonzero quadratic polynomial $f(x, y)$ with $f(x_1, y_1) = \dots = f(x_6, y_6) = 0$.
- For any 6 numbers z_1, z_2, \dots, z_6 , there is a quadratic polynomial $f(x, y)$ with $f(x_i, y_i) = z_i$.

Solution Consider the linear map which sends the vector $(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02})$ to the vector $(f(x_1, y_1), f(x_2, y_2), f(x_3, y_3), f(x_4, y_4), f(x_5, y_5), f(x_6, y_6))$. This is a linear map from $\mathbb{R}^6 \rightarrow \mathbb{R}^6$. Therefore, exactly one of the following happens:

- (1) This map has a nontrivial kernel or
- (2) This map is surjective.

The first situation precisely says that there is a quadratic polynomial through the points (x_i, y_i) and the second precisely says that every function on these points can be interpolated by a quadratic.