2.4.1 One could do this systematically by row reduction, but it is quicker to just fool around:

$$e_4 = \tfrac{1}{2}\alpha_4$$

$$e_3 = \alpha_2 - e_4 = \alpha_2 - \tfrac{1}{2}\alpha_4$$

$$e_1 = \alpha_3 - 4e_4 = \alpha_3 - 2\alpha_4$$

$$e_2 = \alpha_1 - e_1 = \alpha_1 - \alpha_3 + 2\alpha_4$$

So the coordinates of the standard basis vectors in this basis are

$$e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix} \qquad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}.$$

2.4.3 We have $e_1 = \alpha_3$, $e_3 = \alpha_3 - \alpha_1$ and $e_2 = \alpha_2 - e_1 - e_3 = \alpha_2 - \alpha_3 - (\alpha_3 - \alpha_1) = \alpha_1 + \alpha_2 - 2\alpha_3$. So

$$[e_1]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad [e_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad [e_3]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathcal{B}} = [ae_1 + be_2 + ce_3]_{\mathcal{B}} = a\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b-c \\ b \\ a-2b+c \end{bmatrix}.$$

2.4.5 Since $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$, the vectors are not 0. Since $x_1y_1 + x_2y_2 = 0$, the vectors are not proportional.

We now find the coordinates of $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ in the basis $\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right]$. So we want to find scalars c_1 and c_2 with

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

One can do this by slogging ahead in the standard way, but the slick way is to take the dot product of each side of this equation with each of the vectors $\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right]$

$$(x_1a + y_1b) = c_1(x_1^2 + y_1^2) + c_2(x_1x_2 + y_1y_2) = c_1 \cdot 1 + c_2 \cdot 0 = c_1$$

$$(x_2a + y_2b) = c_1(x_1x_2 + y_1y_2) + c_2(x_2^2 + y_2^2) = c_1 \cdot 0 + c_2 \cdot 1 = c_2$$

So

$$\begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} x_1a + y_1b \\ x_2a + y_2b \end{bmatrix}.$$

2.4.6 (a) We want to assume there are no scalars a, b and c such that $a + be^{ix} + ce^{-ix}$ for all x. There are many ways to approach this, one of the easy ones is to plug in $x = 0$, $\pi/2$ and π, to give

$$\begin{array}{ccccccc} a & + & b & + & c & = & 0 \\ a & + & ib & - & ic & = & 0 \\ a & - & b & - & c & = & 0 \end{array}$$

Row reduction then gives $a = b = c = 0$.

2.6.3 The nicest way to do this is to use column reduction to find a particularly nice basis of this space. Here is the effect of the column reduction process:

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & 4 & 4 \\ 1 & -2 & 0 \\ 2 & 5 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1/4 & 0 \\ -2 & 11/4 & 0 \end{bmatrix}.$$

So a basis for the space is $\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/4 \\ 11/4 \end{bmatrix}$. Every vector in this space is thus of the form

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1/4 \\ 11/4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_1+x_2/4 \\ -2x_1+11/4x_2 \end{bmatrix}.$$

So a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is of this form if and only if $x_3 = -x_1 + x_2/4$ and $x_4 = -2x_1 + 11/4x_2$. In other words, we are looking at the vectors which are in the kernel of

$$\begin{bmatrix} 1 & -1/4 & 1 & 0 \\ 2 & -11/4 & 0 & 1 \end{bmatrix}.$$

There are many other ways to do this as well, and many other matrices with the same kernel.

2.6.6 This is very similar to 2.6.3. We start by column reducing

$$\begin{bmatrix} 3 & 1 & 2 & 6 \\ 21 & 7 & 14 & 42 \\ 0 & -1 & 0 & -1 \\ 9 & -2 & 6 & 13 \\ 0 & -1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Our basis is

$$\begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) We have

$$c_1 \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ 7c_1 \\ c_2 \\ 3c_1 + 5c_2 \\ c_3 \end{bmatrix}.$$

So a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is of this form if and only if $x_2 = 7x_1$ and $x_4 = 3x_1 + 5x_3$.

(c) From the above computations, the answer is $\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$.

Problem 1: We have $\mathbb{R}^3 = X + Y$, since any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is of the form $\begin{bmatrix} x \\ y \\ x+y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z-x-y \end{bmatrix}$. To check that $X \cap Y = \{\vec{0}\}$, suppose that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the intersection. Then $x = y = 0$, but also $z = x + y$, so $z = 0$.

Problem 2:

(1) This is true. Every polynomial $f(x)$ can be written as $f(0)+(f(x)-f(0))$, with $f(0) \in C$ and $f(x) - f(0) \in P$, and $C \cap P$ is clearly $\{0\}$.

(2) This is false. If $f(0) \neq f(1)$ then $f(x)$ is not in $C + Q$.

(3) This is false. The polynomial x is in $L \cap P$.

(4) This is true. For a polynomial f, let $\lambda(x)$ be the unique linear polynomial with $\lambda(0) = f(0)$ and $\lambda(1) = f(1)$. Then $f(x) = \lambda(x) + (f(x) - \lambda(x))$, with $\lambda(x) \in L$ and $f(x) - \lambda(x) \in Q$. To see that $L \cap Q = \{0\}$, note that any polynomial of degree ≤ 1 which vanishes at both 0 and 1 is the 0-polynomial.

Problem 3: Recall that any linearly independent subset of V has cardinality at most $\dim V$. Thus, we can find a linearly independent subset S of V which is of maximum cardinality among all S containing $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$. We claim that S is a basis of V. By construction, S is linearly independent, so we just need to check that S spans. Suppose, for the sake of contradiction, that $\vec{w}$ is not in the span of S. Then $S \cup \{\vec{w}\}$ is a larger linearly independent set containing $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

Problem 4

(1) This follows directly from the definitions. The statement $I_n \supseteq I_{n+1}$ says that $\text{Image}(T^n) \supseteq \text{Image}(T^{n+1})$. Indeed, if $\vec{y} = T^{n+1}\vec{x}$, then $\vec{y} = T^n(T\vec{x})$, so any vector in $\text{Image}(T^{n+1})$ is also in $\text{Image}(T^n)$. Similarly, the statement that $K_n \subseteq K_{n+1}$ says that $\text{Ker}(T^n) \subseteq \text{Ker}(T^{n+1})$. Indeed, if $T^n\vec{x} = \vec{0}$ then $T^{n+1}\vec{x} = T(\vec{0}) = \vec{0}$.

(2) From part (1), we have $\dim I_1 \geq \dim I_2 \geq \dim I_3 \geq \cdots$ and $\dim K_1 \leq \dim K_2 \leq \dim K_3 \leq \cdots$. But a decreasing sequence of nonnegative integers must eventually stop, and an increasing sequence of integers bounded above by $\dim V$ must stop. Once the dimensions become equal, the spaces must be equal.

(3) We have $T(I_N) = I_{N+1} = I_N$. So T is a surjective map from $I_N \to I_N$ and, as I_N is a finite dimensional vector space, this means that T restricted to I_N is invertible.

(4) We first check that $I_N \cap K_N = 0$. Let $\vec{x} \in I_N \cap K_N$. So $T^N\vec{x} = \vec{0}$. But $\vec{x} \in I_N$ and we just showed that T restricted to I_N is invertible, so this shows that $\vec{x} = T^{-N}\vec{0} = \vec{0}$.

We now show how to write an arbitrary vector $\vec{v}$ as the sum of a vector from I_N and vector from K_N. The vector $T^N\vec{v}$ is in I_N so, by part (3), we have some vector $\vec{u}$ in I_N with $T^N\vec{u} = T^N\vec{v}$. Then $\vec{v} = \vec{u} + (\vec{v} - \vec{u})$. We constructed $\vec{u}$ to be in I_N, so we will be done if we show that $\vec{v} - \vec{u}$ is in K_N. Indeed, $T^N(\vec{v} - \vec{u}) = T^N\vec{v} - T^N\vec{u} = \vec{0}$.