3.2.6 The image of UT is contained in the image of U. Since U is a map $\mathbb{R}^2 \to \mathbb{R}^3$, its image is at most two dimensional. So the image of UT is at most 2-dimensional, and UT is not invertible. (One can make a similar argument regarding $\mathrm{Ker}(UT)$.) More generally, if $p < q$, T is a map $\mathbb{R}^q \to \mathbb{R}^p$ and U is a map $\mathbb{R}^p \to \mathbb{R}^q$, then the composition UT is not invertible.

3.2.8 The condition that $T^2 = 0$ means that, for any vector $\vec{v}$, the image $T(\vec{v})$ is in $\mathrm{Ker}(T)$. Thus, $\mathrm{Im}(T) \subseteq \mathrm{Ker}(T)$. An example of a linear operator for which $T^2 = 0$ but $T \neq 0$ is $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

3.4.8 Following the hint, let $T = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$. We want to find a vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ with $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = e^{i\theta}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. We write this out as equations:

$$\begin{array}{ccccc} (\cos\theta)x & - & (\sin\theta)y & = & e^{i\theta}x \\ (\sin\theta)x & + & (\cos\theta)y & = & e^{i\theta}y \end{array}.$$

Using the relations $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, this gives:

$$\begin{array}{ccccc} \frac{-e^{i\theta}+e^{i\theta}}{2}x & - & \frac{e^{i\theta}-e^{-i\theta}}{2i}y & = & 0 \\ \frac{e^{i\theta}-e^{-i\theta}}{2i}x & + & \frac{-e^{i\theta}+e^{i\theta}}{2}y & = & 0 \end{array}.$$

These equations give

$$\frac{y}{x} = \frac{(-e^{i\theta}+e^{i\theta})/2}{(e^{i\theta}-e^{-i\theta})/(2i)} = -i \text{ and } \frac{y}{x} = \frac{(e^{i\theta}-e^{-i\theta})/(2i)}{(e^{i\theta}-e^{i\theta})/2} = -i.$$

Fortunately, these equations are consistent, and they show that we can take $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right]$. Similarly, we have $T\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right] = e^{-i\theta}\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]$.

So

$$T = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}.$$

3.5.2 Let the dual basis be β_1, β_2, β_3. So we must have $\alpha_i \cdot \beta_i = 1$ and $\alpha_i \cdot \beta_j = 0$. We can write these equations conveniently in matrix form

$$\begin{bmatrix} - & \alpha_1 & - \\ - & \alpha_2 & - \\ - & \alpha_3 & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \beta_1 & \beta_2 & \beta_3 \\ | & | & | \end{bmatrix} = \mathrm{Id}_3$$

or, concretely,

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} | & | & | \\ \beta_1 & \beta_2 & \beta_3 \\ | & | & | \end{bmatrix} = \mathrm{Id}_3$$

So

$$\begin{bmatrix} | & | & | \\ \beta_1 & \beta_2 & \beta_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1/2 \\ -1 & -1 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}.$$

So

$$\beta_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \beta_3 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}.$$

3.5.8 A vector $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}$ is in $W^\perp$ (or, as the book says, W°) if it satisfies the equations:

$$\begin{array}{ccccccccccc} x_1 & + & 2x_2 & + & x_3 & & & & & = & 0 \\ & & x_2 & + & 3x_3 & + & 3x_4 & + & x_5 & = & 0 \\ x_1 & + & 4x_2 & + & 6x_3 & + & 4x_4 & + & x_5 & = & 0 \end{array} .$$

Row reducing these equations, we get

$$\begin{array}{ccccccccc} x_1 & & & & 4x_4 & + & 3x_5 & = & 0 \\ & x_2 & & - & 3x_4 & - & 2x_5 & = & 0 \\ & & x_3 & + & 2x_4 & + & x_5 & = & 0 \end{array}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_4 - 3x_5 \\ 3x_4 + 2x_5 \\ -2x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

So a basis for $W^\perp$ is

$$\begin{bmatrix} -4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

(Of course, there are many other bases.)

Problem 1. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Let $f_1 = e_1$, $f_2 = e_2$ and $f_3 = e_1+e_2+e_3$. Express the dual basis vectors f_1^*, f_2^* and f_3^* as a linear combination of e_1^*, e_2^* and e_3^*. You should find that, even though $e_1 = f_1$ and $e_2 = f_2$, the dual vectors f_1^* and f_2^* are different from e_1^* and e_2^*.

Solution: Writing $f_1^* = (ae_1^* + be_2^* + ce_3^*)$, we must have

$$\begin{array}{lcl} f_1^*(f_1)=1 & \Longrightarrow & a =1 \\ f_1^*(f_2)=0 & \Longrightarrow & \phantom{a+{}} b =0 \\ f_1^*(f_3)=0 & \Longrightarrow & a+b+c=0 \end{array} .$$

Solving the equations, $a = 1$, $b = 0$ and $c = -1$, so $f_1^* = e_1^* - e_3^*$. Similarly, $f_2^* = e_2^* - e_3^*$ and $f_3^* = e_3^*$.

Problem 2. Let $\vec{v}_1$,$\vec{v}_2$, $\dots$, $\vec{v}_n$ and $\vec{w}_1$, $\vec{w}_2$, $\dots$, $\vec{w}_n$ be two bases of a vector space V, and let v_1^*, v_2^*, $\dots$, v_n^* and w_1^*, w_2^*, $\dots$, w_n^* be the dual bases. Let the matrices A and B be defined by $\vec{w}_j = \sum_i A_{ij}\vec{v}_i$ and $w_j^* = \sum_i B_{ij}\vec{v}_i^*$. Show that $B = (A^T)^{-1}$.

Solution: We are supposed to have

$$w_j^*(\vec{w}_\ell) = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases} .$$

Expanding the left hand side, we have

$$w_j^*(\vec{w}_\ell) = \left(\sum_i B_{ij} v_i^* \right) \left(\sum_k A_{k\ell} \vec{v}_k \right) = \sum_{i,k} B_{ij} A_{k\ell} v_i^*(\vec{v}_k) = \sum_k B_{kj} A_{k\ell}.$$

In the last step, we have used that $v_i^*(\vec{v}_k) = 1$ if $i = k$, and $v_i^*(\vec{v}_k) = 0$ for $i \neq k$.

But $\sum_k B_{kj} A_{k\ell}$ is the (j, ℓ) entry of $B^T A$. So we have shown that $B^T A = \mathrm{Id}$ and hence $B = (A^T)^{-1}$.

Problem 3. Let C be the vector space of real polynomials of degree ≤ 3. For a real number r, let a_r be the function $f(x) \mapsto f(r)$ in C^*.

(1) Show that, if r_1, r_2, r_3, r_4 are four distinct real numbers, then a_{r_1}, a_{r_2}, a_{r_3}, a_{r_4} is a basis of C^*.
(2) Express the linear function $\int_0^3 f(x)dx$ as a linear combination of a_0, a_1, a_2 and a_3.

Solution (1) Since C is 4-dimensional, so is C^*, so it is enough to either show that a_{r_1}, a_{r_2}, a_{r_3}, a_{r_4} span, or to show that they are linearly independent. I'll check linear independence.

Suppose we had a linear relation $c_1 a_{r_1} + c_2 a_{r_2} + c_3 a_{r_3} + c_4 a_{r_4} = 0$. Concretely, this means that, for every cubic polynomial $f(x)$, we have

$$c_1 f(r_1) + c_2 f(r_2) + c_3 f(r_3) + c_4 f(r_4) = 0.$$

Taking $f(x) = (x - r_2)(x - r_3)(x - r_4)$, we get

$$c_1 (r_1 - r_2)(r_1 - r_3)(r_2 - r_4) + 0 + 0 + 0 = 0$$

so $c_1 = 0$. Similar, $c_2 = c_3 = c_4 = 0$ and we have proved linear independence.

(2) We want to find coefficients c_0, c_1, c_2, c_3 such that, for all cubics $f(x)$, we have

$$\int_0^3 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2) + c_3 f(3).$$

Plugging in the cubic $(x - 0)(x - 1)(x - 2)$, we get

$$\int_0^3 (x-0)(x-1)(x-2)dx = 6c_3$$

so

$$c_3 = \tfrac{1}{6} \int_0^3 (x-0)(x-1)(x-2)dx = \tfrac{1}{6}\tfrac{9}{4} = \tfrac{3}{8}.$$

Similarly

$$c_0 = -\tfrac{1}{6} \int_0^3 (x-1)(x-2)(x-3)dx = (\tfrac{-1}{6})(\tfrac{-9}{4}) = \tfrac{3}{8}.$$

$$c_1 = \tfrac{1}{2} \int_0^3 (x-0)(x-2)(x-3)dx = (\tfrac{1}{2})(\tfrac{9}{4}) = \tfrac{9}{8}.$$

$$c_2 = \tfrac{1}{2} \int_0^3 (x-0)(x-1)(x-3)dx = (\tfrac{1}{2})(\tfrac{9}{4}) = \tfrac{9}{8}.$$

So the integral is $\frac{3}{8}a_0 + \frac{9}{8}a_1 + \frac{9}{8}a_2 + \frac{3}{8}a_3$.

We have shown that, for cubic polynomials, we have

$$\int_0^3 f(x)dx = \tfrac{3}{8}\left(f(0) + 3f(1) + 3f(2) + f(3)\right).$$

It also turns out for other smooth functions $f(x)$, we have the excellent approximation:

$$\int_0^3 f(x)dx \approx \tfrac{3}{8}\left(f(0) + 3f(1) + 3f(2) + f(3)\right).$$

This is called "Simpson's 3/8 rule".