Solution Set Six

5.2.3 These are all straightforward computations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}.$$

$$\det \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = da - (-b)(-c) = ad - bc.$$

And

$$\operatorname{adj} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^T.$$

5.2.4 If $AB = \mathrm{Id}$ then $\det(AB) = \det(A)\det(B)$, so $\det(A) \neq 0$. When $\det(A) \neq 0$, we have the explicit formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

5.2.9 (a) By linearity, we have $D(\vec{v}_1, \vec{v}_2, \ldots, 0, \ldots, \vec{v}_n) = 0D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, \vec{v}_n) = 0$, where the vector $\vec{v}_i$ may be chosen arbitrarily.

(b) Suppose we add a scalar multiple of $\vec{v}_i$ to $\vec{v}_j$. Then

$$D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, c\vec{v}_i + \vec{v}_j, \ldots, \vec{v}_n) = cD(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, \vec{v}_i, \ldots, \vec{v}_n) + D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots, \vec{v}_n) = c \cdot 0 + D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots, \vec{v}_n).$$

We have used the alternating property to show that $D(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i, \ldots, \vec{v}_i, \ldots, \vec{v}_n) = 0$.

5.2.10 (a) The rank of a 2×3 matrix is at most 2. If $c_1 \neq 0$, then the second and third columns of A are linearly independent, so $\operatorname{rank}(A) = 2$; a similar argument applies if c_2 or $c_3 \neq 0$. Conversely, suppose that $\operatorname{rank}(A) = 2$, so the image of A is two dimensional. The image of A is spanned by the columns of A, so there must be two linearly independent columns of A, and those two columns give a nonzero c_i.

(b) Assuming that A has rank 2, the kernel of A is one dimensional, so we just need to check that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words, we need to check that

$$a_{11}(a_{12}a_{23} - a_{13}a_{22}) + a_{12}(a_{13}a_{21} - a_{11}a_{23}) + a_{13}(a_{11}a_{22} - a_{12}a_{21}) = 0 \text{ and}$$

$$a_{21}(a_{12}a_{23} - a_{13}a_{22}) + a_{22}(a_{13}a_{21} - a_{11}a_{23}) + a_{23}(a_{11}a_{22} - a_{12}a_{21}) = 0.$$

This is a straightforward computation.

5.3.7. If A is upper triangular, this means that $A_{ij} = 0$ whenever $i > j$. The determinant of A is the sum of $\pm A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$, where the sum ranges over all permutations σ of $\{1, 2, 3, \ldots, n\}$. If σ is a permutation which is not the identity, then there is some i with $\sigma(i) < i$, so $A_{i\sigma(i)} = 0$. So the determinant is $A_{11}A_{22} \cdots A_{nn}$.

Problem 1. Let V and W be vector spaces and let $A : V \to W$ be a linear transformation.

(1) Show that $\operatorname{Ker}(A^*) = \operatorname{Im}(A)^\perp$.
(2) Show that, if V and W are finite dimensional, we also have $\operatorname{Im}(A^*) = \operatorname{Ker}(A)^\perp$.

Solution: (1) Let w^* be a vector in W^*. By definition, w^* is in $\operatorname{Ker}(A^*)$ if $A^*(w^*) = 0$, which is the same as saying that, for every $\vec{v} \in V$, we have $A^*(w^*)(\vec{v}) = 0$. Unwinding the definition of A^*, this is the same as saying that $w^*(A(\vec{v})) = 0$ for all $\vec{v} \in V$. Meanwhile, the definition of $\operatorname{Ker}(A)^\perp$ is also that $w^*(A(\vec{v})) = 0$ for all $\vec{v} \in V$.

(2) We can show that $\mathrm{Im}(A^*) \subseteq \mathrm{Ker}(A)^\perp$ without using finite dimensionality. Indeed, let $w^* \in W^*$, we will show that $A^*(w^*)$ is in $\mathrm{Ker}(A)^\perp$. In other words, we must show that, if $\vec{v} \in \mathrm{Ker}(A)$, then $A^*(w^*)(\vec{v}) = 0$. Indeed, we have $A^*(w^*)(\vec{v}) = w^*(A(\vec{v}) = w^*(0) = 0$.

Now, for the reverse direction, we compute dimensions. Let $\dim V = m$, let $\dim W = n$ and let r be the rank of A. Then A^* also has rank r. So $\dim \mathrm{Im}(A^*) = r$, while $\dim \mathrm{Ker}(A) = m - r$, so $\dim \mathrm{Ker}(A)^\perp = m-(m-r) = r$. We showed in the previous paragraph that $\mathrm{Im}(A^*) \subseteq \mathrm{Ker}(A)^\perp$, and we have now shown that they both have the same dimension, so they are equal.

Problem 2. Let V be a vector space over $\mathbb{R}$, and let $A : V \times V \times V \longrightarrow \mathbb{R}$ be an alternating multilinear form. Let $\vec{x}$, $\vec{y}$, $\vec{z}$ be three vectors in V with $A(\vec{x}, \vec{y}, \vec{z}) = 17$. Compute the following, directly using the axioms of an alternating form:

(1) $A(\vec{y}, \vec{z}, \vec{x})$.
(2) $A(\vec{x}, 2\vec{x} + 3\vec{y}, 4\vec{x} + 5\vec{y} + 6\vec{z})$.
(3) $A(\vec{x} + 2\vec{y} + 3\vec{z}, 4\vec{x} + 5\vec{y}, 6\vec{x})$.
(4) $A(2\vec{x} + \vec{y}, \vec{x} + 2\vec{y}, \vec{z})$.

Solution

$$A(\vec{y}, \vec{z}, \vec{x}) = -A(\vec{y}, \vec{x}, \vec{z}) = A(\vec{x}, \vec{y}, \vec{z}) = 17.$$

$$A(\vec{x}, 2\vec{x} + 3\vec{y}, 4\vec{x} + 5\vec{y} + 6\vec{z}) = A(\vec{x}, 3\vec{y}, 5\vec{y} + 6\vec{z}) = \\ 3A(\vec{x}, \vec{y}, 5\vec{y} + 6\vec{z}) = 3A(\vec{x}, \vec{y}, 6\vec{z}) = 18A(\vec{x}, \vec{y}, \vec{z}) = 18 \times 17 = 306.$$

$$A(\vec{x} + 2\vec{y} + 3\vec{z}, 4\vec{x} + 5\vec{y}, 6\vec{x}) = 6A(\vec{x} + 2\vec{y} + 3\vec{z}, 4\vec{x} + 5\vec{y}, \vec{x}) = 6A(2\vec{y} + 3\vec{z}, 5\vec{y}, \vec{x}) = \\ 6 \times 5 \times A(2\vec{y}+3\vec{z}, \vec{y}, \vec{x}) = 6 \times 5 \times A(3\vec{z}, \vec{y}, \vec{x}) = 6 \times 5 \times 3A(\vec{z}, \vec{y}, \vec{x}) = -6 \times 5 \times 3 \times 17 = -1530.$$

$$A(2\vec{x}+\vec{y}, \vec{x}+2\vec{y}, \vec{z}) = 2A(\vec{x}, \vec{x}, \vec{z})+4A(\vec{x}, \vec{y}, \vec{z})+A(\vec{y}, \vec{x}, \vec{z})+2A(\vec{y}, \vec{y}, \vec{z}) = 2\times0+4\times17-17+0\times17 = 51.$$

Problem 3. Let V be a vector space of dimension n over a field F. Let $A : V \times V \times V \longrightarrow F$ be a multilinear form. We will say that A is ***symmetric*** if, for all vectors $\vec{u}$, $\vec{v}$, $\vec{w} \in V$, we have

$$A(\vec{u}, \vec{v}, \vec{w}) = A(\vec{u}, \vec{w}, \vec{v}) = A(\vec{v}, \vec{u}, \vec{w}) = A(\vec{v}, \vec{w}, \vec{u}) = A(\vec{w}, \vec{u}, \vec{v}) = A(\vec{w}, \vec{v}, \vec{u}).$$

What is the dimension of the vector space of symmetric bilinear forms $A : V \times V \times V \longrightarrow F$?

A bilinear form is determined by its values on the triples of basis vectors, meaning the values $A(e_i, e_j, e_k)$ for $1 \leq i, j, k \leq n$. Moreover, by symmetry, we can reorder the inputs, so A is determined by the values of $A(e_i, e_j, e_k)$ for $1 \leq i \leq j \leq k \leq n$. Conversely, for any values we assign to those $A(e_i, e_j, e_k)$, we can define a symmetric trilinear form. So we just need to count the number of triples (i, j, k) with $1 \leq i \leq j \leq k \leq n$. The number of such triples is $\frac{n(n+1)(n+2)}{6}$.

Problem 4. Let H be the vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(0) = f(1) = 0$. For $f(x)$ and $g(x)$ in H, define

$$\langle f, g \rangle = \int_0^1 f(x)g'(x)dx.$$

Show that $\langle\ ,\ \rangle$ is an alternating bilinear form $H \times H \to \mathbb{R}$. (You need to check both that it is bilinear and that it is alternating.)

We check bilinearity:

$$\int_0^1 (f_1(x) + f_2(x))g'(x)dx = \int_0^1 f_1(x)g'(x)dx + \int_0^1 f_2(x)g'(x)dx.$$

$$\int_0^1 f(x)\tfrac{d(g_1(x)+g_2(x))}{dx}dx = \int_0^1 f(x)(g_1'(x) + g_2'(x))dx = \int_0^1 f(x)g_1'(x)dx + \int_0^1 f(x)g_2'(x)dx.$$

$$\int_0^1 (cf(x))g'(x)dx = c\int_0^1 f(x)g'(x)dx = \int_0^1 f(x)\tfrac{d(cg(x))}{dx}dx.$$

Now, we check that the form is alternating:

$$\int_0^1 f(x)f'(x)dx = \tfrac{1}{2}f(x)\Big|_{x=0}^{1} = \tfrac{1}{2}\left(f(1)^2 - f(0)^2\right) = \tfrac{1}{2}\left(0^2 - 0^2\right) = 0.$$