Problem 1. Recall that the cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - y_2x_3 \\ x_3y_1 - y_3x_1 \\ x_1y_2 - y_1x_2 \end{bmatrix}.$$

For any vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ in $\mathbb{R}^3$, define $B_{\vec{c}}(\vec{x}, \vec{y}) := \vec{c} \cdot (\vec{x} \times \vec{y})$.

(1) Show that, for any vector $\vec{c} \in \mathbb{R}^3$, the function $B_{\vec{c}}(\ , \ )$ is an alternating bilinear form.
(2) Let $B(\ , \ )$ be any alternating bilinear form on $\mathbb{R}^3$. Show that there is a unique vector $\vec{c} \in \mathbb{R}^3$ such that $B(\ , \ )$ is $B_{\vec{c}}(\ , \ )$.

Solution: Part 1: We must check:

- Additivity: $\vec{c} \cdot ((\vec{x}_1 + \vec{x}_2) \times \vec{y}) = \vec{c} \cdot (\vec{x}_1 \times \vec{y} + \vec{x}_2 \times \vec{y}) = \vec{c} \cdot (\vec{x}_1 \times \vec{y}) + \vec{c} \cdot (\vec{x}_2 \times \vec{y})$ and $\vec{c} \cdot (\vec{x} \times (\vec{y}_1 + \vec{y}_2)) = \vec{c} \cdot (\vec{x} \times \vec{y}_1 + \vec{x} \times \vec{y}_2) = \vec{c} \cdot (\vec{x} \times \vec{y}_1) + \vec{c} \cdot (\vec{x} \times \vec{y}_2)$.
- Scalar multiplication: $\vec{c} \cdot ((a\vec{x}) \times \vec{y}) = a\vec{c} \cdot (\vec{x} \times \vec{y}) = \vec{c} \cdot (\vec{x} \times (a\vec{y}))$.
- Alternation: $\vec{c} \cdot (\vec{x} \times \vec{x}) = \vec{c} \cdot \vec{0} = 0$.

Part 2: We first write down a general alternating bilinear form. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$. Then $B(e_1, e_1) = B(e_2, e_2) = B(e_3, e_3) = 0$ and define $B(e_1, e_2) = -B(e_2, e_1) = p$, $B(e_1, e_3) = -B(e_3, e_1) = r$, $B(e_2, e_3) = -B(e_3, e_2) = r$. Then we have

$$\begin{aligned} B(x_1e_1 + x_2e_2 + x_3e_3, \ y_1e_1 + y_2e_2 + y_3e_3) &= \sum_{i,j=1}^{3} x_i y_j B(e_i, e_j) \\ &= p(x_1y_2 - x_2y_1) + q(x_1y_3 - x_3y_1) + r(x_2y_3 - x_3y_2). \end{aligned}$$

This is $\begin{bmatrix} r \\ -q \\ p \end{bmatrix} \cdot (\vec{x} \times \vec{y})$, so we take $\vec{c} = \begin{bmatrix} r \\ -q \\ p \end{bmatrix}$ (and no other $\vec{c}$ works).

Problem 2. Let V be an n dimensional vector space over a field F. Let e_1, e_2, $\dots$, e_n be one basis for V and let f_1, f_2, $\dots$, f_n be another basis. Let S be the matrix defined by $f_j = \sum_i S_{ij} e_i$.

(1) Let $T : V \to V$ be a linear map and define the matrices X and Y by $T(e_j) = \sum_i X_{ij} e_i$ and $T(f_j) = \sum_i Y_{ij} f_i$. Give a formula for Y in terms of X and S.
(2) Show that $\det X = \det Y$.
(3) Let $B : V \times V \longrightarrow F$ be a bilinear form and define the matrices P and Q by $B(e_i, e_j) = P_{ij}$ and $B(f_i, f_j) = Q_{ij}$. Give a formula for Q in terms of P and S.
(4) Show that there is a nonzero element $s \in F$ with $\det P = s^2 \det Q$.

Solution Part 1: There are slicker ways to do this computation, but I'll write out the brute force solution. We have

$$T(f_j) = \sum_i Y_{ij} f_i = \sum_i Y_{ij} \left(\sum_h S_{hi} e_h \right) = \sum_{\ell, i} S_{\ell i} Y_{ij} e_\ell.$$

But also

$$T(f_j) = T\left(\sum_k S_{kj} e_k \right) = \sum_k S_{kj} T(e_k) = \sum_k S_{kj} \left(\sum_\ell X_{\ell k} e_\ell \right) = \sum_{k, \ell} X_{\ell k} S_{kj} e_\ell.$$

Since the e's are a basis, we can set their coefficients equal to get

$$\sum_i S_{\ell i} Y_{ij} = \sum_k X_{\ell k} S_{kj}.$$

Written as a matrix equation, this says that $SY = XS$, so $Y = S^{-1}XS$.

Part 2: Taking determinants, we get $\det(Y) = \det(S)^{-1}\det(X)\det(S) = \det(X)$.

Part 3: We compute

$$Q_{ij} = B(f_i, f_j) = B\left(\sum_h S_{hi}e_h, \sum_k S_{kj}e_k\right) = \sum_{h,k} S_{hi}S_{kj}B(e_h, e_k) = \sum_{h,k} S_{hi}S_{kj}P_{hk}.$$

We can write this as a matrix equation: $Q = S^T PS$.

Part 4: We have $\det(Q) = \det(S^T)\det(P)\det(S) = \det(S)^2\det(P)$ where $s = \det(S)$.

Problem 3. Let V be an n-dimensional vector space over a field F and let $B : V \times V \to F$ be an alternating bilinear form. In this problem, we will show that there is some integer r such that there is a basis $\vec{x}_1$, $\vec{y}_1$, $\vec{x}_2$, $\vec{y}_2$, $\dots$, $\vec{x}_r$, $\vec{y}_r$, $\vec{z}_1$, $\vec{z}_2$, $\dots$, $\vec{z}_{n-2r}$ of V such that $B(\vec{x}_i, \vec{y}_i) = -B(\vec{y}_i, \vec{x}_i) = 1$ and all other pairings between the basis vectors are 0. This proof is by induction on n.

(1) Do the base cases $n = 1$ and $n = 2$.
(2) Explain why we are done if $B(\vec{v}, \vec{w}) = 0$ for all vectors $\vec{v}$ and $\vec{w}$ in V.

From now on, assume that $n > 2$ and that $B(\vec{v}, \vec{w})$ is not always 0. Choose two vectors $\vec{x}$, $\vec{y}$ with $B(\vec{x}, \vec{y}) = 1$. Set $V' = \{\vec{v} : B(\vec{x}, \vec{v}) = B(\vec{y}, \vec{v}) = 0\}$.

(3) Show that $V = \mathrm{Span}(\vec{x}, \vec{y}) \oplus V'$.
(4) By induction, V' has a basis $\vec{x}_1$, $\vec{y}_1$, $\vec{x}_2$, $\vec{y}_2$, $\dots$, $\vec{x}_r$, $\vec{y}_r$, $\vec{z}_1$, $\vec{z}_2$, $\dots$, $\vec{z}_{n-2-2r}$ as required. Explain how to finish the proof from here.
(5) We conclude with an example. Consider the alternating bilinear form

$$B((u_1, u_2, u_3, u_4),\ (v_1, v_2, v_3, v_4)) = \sum_{1 \le i < j \le 4} (u_i v_j - u_j v_i)$$

on $\mathbb{R}^4$. Find a basis $\vec{x}_1$, $\vec{y}_1$, $\vec{x}_2$, $\vec{y}_2$ as above.

Solution Part 1: When $n = 1$, the form B must be 0, since there is only one basis vector e_1 and we have $B(e_1, e_1) = 0$. So we take $r = 0$ and $\vec{z}_1 = e_1$. Now, take $n = 2$. Let e_1, e_2 be a basis for V. If $B(e_1, e_2) = 0$, then B is 0, so we can take $r = 0$ with $\vec{z}_1 = e_1$ and $\vec{z}_2 = e_2$. If $B(e_1, e_2) = b \neq 0$, then we can take $r = 1$ with $\vec{x}_1 = e_1$ and $\vec{y}_1 = e_2/b$.

Part 2: We just take $r = 0$ and take $\vec{z}_1$, $\vec{z}_2$, $\dots$, $\vec{z}_n$ to be any basis of V.

Part 3: We need to check that $\mathrm{Span}(\vec{x}, \vec{y}) \cap V' = \{0\}$ and $V = \mathrm{Span}(\vec{x}, \vec{y}) + V'$. For the first, consider a vector $a\vec{x}+b\vec{y}$. This vector will be in V' if and only if $B(\vec{x}, a\vec{x}+b\vec{y}) = B(\vec{y}, a\vec{x}+b\vec{y}) = 0$. We expand $B(\vec{x}, a\vec{x} + b\vec{y}) = bB(\vec{x}, \vec{y}) = b$ and $B(\vec{y}, a\vec{x} + b\vec{y}) = aB(\vec{y}, \vec{x}) = -a$. So $-a = b = 0$ and the vector is 0.

Now, we must show that any vector $\vec{v} \in V$ is of the form $(a\vec{x} + b\vec{y}) + \vec{w}$ for $\vec{w} \in V'$. Indeed, take $\vec{w} = \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}$. We need to check that $\vec{w}$ is in V'. We have

$$B(\vec{x}, \vec{w}) = B\big(\vec{x}, \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}\big) = B(\vec{x}, \vec{v}) - 0 - B(\vec{x}, \vec{v})B(\vec{x}, \vec{y}) = B(\vec{x}, \vec{v}) - B(\vec{x}, \vec{v})1 = 0.$$

$$B(\vec{y}, \vec{w}) = B\big(\vec{y}, \vec{v} - B(\vec{v}, \vec{y})\vec{x} - B(\vec{x}, \vec{v})\vec{y}\big) = B(\vec{y}, \vec{v}) - B(\vec{v}, \vec{y})B(\vec{y}, \vec{x}) + 0 = B(\vec{y}, \vec{v}) - (-1)B(\vec{v}, \vec{y}) = 0.$$

Part 4: We simply take the basis $\vec{x}_1$, $\vec{y}_1$, $\vec{x}_2$, $\vec{y}_2$, $\dots$, $\vec{x}_r$, $\vec{x}$, $\vec{y}$, $\vec{z}_1$, $\vec{z}_2$, $\dots$, $\vec{z}_{n-2-2r}$. We constructed V' such that $B(\vec{x}, \vec{v}) = B(\vec{y}, \vec{v}) = 0$ for all $\vec{v}$ in V', which means that $\vec{x}$ and $\vec{y}$ have the correct pairing with all the other basis vectors in this list. They also pair to 1 with each other and all the other vectors, inductively, have the correct pairing, so the result holds.

Part 5: We carry out the algorithm implied in the inductive procedure. Put $\vec{x}_1 = (1, 0, 0, 0)$ and $\vec{y}_1 = (0, 1, 0, 0)$. Let $V' = \{\vec{v} : B(\vec{x}_1, \vec{v}) = B(\vec{y}_1, \vec{v}) = 0\}$. We compute explicitly that V' is the set of (v_1, v_2, v_3, v_4) such that

$$v_2 + v_3 + v_4 = -v_1 + v_3 + v_4 = 0.$$

We see that a basis of V' is $(1, -1, 1, 0)$, $(1, -1, 0, 1)$. Then $B\left((1, -1, 1, 0),\ (1, -1, 0, 1)\right) = (-1) + 1 - (-1) + (-1) - 1 - (-1) - 1 - (-1) + 1 = 1$. So we can take $\vec{x}_2 = (1, -1, 1, 0)$ and $\vec{y}_2 = (1, -1, 0, 1)$.