Solution Set Eight

6.2.3 More precisely, we will show that, if A is upper triangular, then the characteristic polynomial of A is $\prod_{i=1}^{n}(x - A_{ii})$. We can prove this by induction on n. Doing row expansion on the bottom row of A gives

$$\det \begin{bmatrix} x-A_{11} & -A_{12} & -A_{13} & \cdots & -A_{1(n-1)} & A_{-1n} \\ 0 & x-A_{22} & -A_{23} & \cdots & -A_{2(n-1)} & A_{-2n} \\ 0 & 0 & x-A_{33} & \cdots & -A_{3(n-1)} & A_{-3n} \\ & & & \ddots & \vdots & \vdots \\ & & & & x-A_{(n-1)(n-1)} & -A_{(n-1)n} \\ & & & & & x-A_{nn} \end{bmatrix}$$

$$= (x - A_{nn}) \det \begin{bmatrix} x-A_{11} & -A_{12} & -A_{13} & \cdots & -A_{1(n-1)} \\ 0 & x-A_{22} & -A_{23} & \cdots & -A_{2(n-1)} \\ 0 & 0 & x-A_{33} & \cdots & -A_{3(n-1)} \\ & & & \ddots & \vdots \\ & & & & x-A_{(n-1)(n-1)} \end{bmatrix}$$

$$= (x - A_{nn}) \prod_{i=1}^{n-1}(x - A_{ii}) = \prod_{i=1}^{n}(x - A_{ii})$$

as desired.

6.2.4 We first compute the characteristic polynomial:

$$\det \begin{bmatrix} x+9 & -4 & -4 \\ 8 & x-3 & -4 \\ 16 & -8 & x-7 \end{bmatrix} = x^3 - x^2 - 5x - 3 = (x+1)^2(x-3).$$

So the eigenvalues are -1 and 3.

We now compute the eigenspaces. The (-1)-eigenspace is the kernel of

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}.$$

This matrix row reduces to

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the (-1)-eigenspace is thus $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$.

The 3-eigenspace is the kernel of

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}.$$

This matrix row reduces to

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So a basis for the 3-eigenspace is $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

In short, a basis of eigenvectors is $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

6.2.10 Let $A = \left[\begin{smallmatrix} p & q \\ q & r \end{smallmatrix}\right]$. So the characteristic polynomial of A is $(x-p)(x-r) - q^2 = x^2 - (p+r)x + (pr - q^2)$. The roots of this polynomial are

$$\frac{p + r \pm \sqrt{(p+r)^2 - 4(pr - q^2)}}{2} = \frac{p + r \pm \sqrt{p^2 - 2pr + r^2 + 4q^2}}{2}.$$

We have $p^2 - 2pr + r^2 + 4q^2 = (p-r)^2 + 4q^2 \geq 0$, so the square root is real.

If $(p-r)^2+4q^2>0$, then we have two distinct real eigenvalues, so the matrix is diagonalizable. If $(p-r)^2+4q^2=0$ then $p=r$ and $q=0$, so $A=\left[\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right]$, which is already diagonal.

6.4.2 Let $m(x)$ be the minimal polynomial of T, so $m(T)=0$. Then $m(T|_W)=m(T)|_W=0|_W=0$. Thus, the minimal polynomial of $T|_W$ must divide $m(x)$.

6.4.5 Let V_0 and V_1 be the 0-eigenspace and the 1-eigenspace of A. We'll show that $V=V_0\oplus V_1$. It is easy to see that $V_0\cap V_1=\{\vec{0}\}$: If $\vec{v}$ is in $V_0\cap V_1$ then $A\vec{v}=0\vec{v}$ and $A\vec{v}=1\vec{v}$, so $\vec{v}=0$.

We need to work harder to see that $V=V_0+V_1$. Let $\vec{v}$ be any vector in V. Then we have $\vec{v}=(\vec{v}-A\vec{v})+A\vec{v}$. We have $A(\vec{v}-A\vec{v})=A\vec{v}-A^2\vec{v}=0$ (since $A^2=A$), so $\vec{v}-A\vec{v}\in V_0$. We similarly have $A(A\vec{v})=A^2\vec{v}=A\vec{v}$, so $A\vec{v}\in V_1$.

6.4.7 If T is diagonalizable, with $\lambda_1, \lambda_2, \ldots, \lambda_r$ the list of distinct eigenvalues, then $\prod(T-\lambda_j \mathrm{Id})=0$. The reverse direction is almost proved by Theorem 6 in the book, but we have a few details to fill in. Let $f(x)=\prod(x-\alpha_i)$ be a polynomial with distinct roots such that $f(T)=0$. Let $m(x)$ be the minimal polynomial of T. Then $m(x)$ divides $f(x)$, so $m(x)$ also factors as a product of distinct linear factors. Then Theorem 6 says that T is diagonalizable.

Problem 1. Let V be a finite dimensional vector space, let $A: V\to V$ be a linear transformation and suppose that U is a subspace of V such that $AU\subseteq U$.

(1) Show that there is a basis of V in which A takes the form $\left[\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right]$.
(2) Show that there is a basis of U such that the restriction $A|_U$ given by the matrix P.
(3) Show that there is a linear map $\bar{A}: V/U \longrightarrow V/U$ defined by $\bar{A}(v+U)=A(V)+U$. (In other words, show that, if $v_1+U=v_2+U$, then $\bar{A}(v_1)+U=\bar{A}(v_2)+U$ and this function $V/U\longrightarrow V/U$ is linear.)
(4) Show that there is a basis for V/U where $\bar{A}$ is given by the matrix R.

Solution: (1) Take a basis $u_1, u_2, \ldots, u_k$ of U and complete it to a basis $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}$ of V. Since $AU\subseteq U$, we have $Au_j\in U$ for each j; write $Au_j=\sum_i P_{ij}u_i$. Then, in the basis $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{n-k}$, the first k columns are of the form $\left[\begin{smallmatrix} P \\ 0 \end{smallmatrix}\right]$.

(2) In the basis u_i for U which we just discussed, the matrix of $A|_U$ is P.

(3) As described in the parenthetical, we first check that, if $v_1+U=v_2+U$, then $Av_1+U=Av_2+U$. Indeed, suppose that $v_2=v_1+u$ for $u\in U$. Then $Av_2=A(v_1+u)=Av_1+Au$, and $Au\in U$, so $Av_2=Av_1+U$ as required. We also want to check that the map is linear. Indeed, $A(v_1+v_2+U)=Av_1+Av_2+U=(Av_1+U)+(Av_2+U)$ and, for any scalar c, we have $A(c(v+U))=A(cv+U)=A(cv)+U=cAv+U=c(A(v+U))$

(4) We take the basis $v_1+U, v_2+U, \ldots, v_{n-k}+U$ for V/U. We have $Av_j=\sum_h Q_{hj}u_h+\sum_i R_{ij}v_i$. Since $\sum_h Q_{hj}u_h\in U$, we have $Av_j+U=\sum_i R_{ij}v_i+U$. We rewrite this as $\bar{A}(v_j+U)=\sum_i R_{ij}(v_i+U)$. So, in the basis $v_1+U, v_2+U, \ldots, v_{n-k}+U$, the linear transformation $\bar{A}$ is given by the matrix R.

Problem 2. For a polynomial f with real coefficients, define $D(f)=xf'+f''$. For each positive integer n, show that there is a polynomial of degree $\leq n$ such that $D(f)=nf$. (Hint: What does this have to do with eigenvalues?)

Let V be the vector space of polynomials of degree $\leq n$. A basis of V is $1, x, x^2, \ldots, x^n$. We have $D(x^k)=x(kx^{k-1})+k(k-1)x^{k-2}=kx^k+k(k-1)x^{k-2}$. So the matrix of D in this basis is

$$\begin{bmatrix} 0 & 0 & 2 & & & & & \\ & 1 & 0 & 6 & & & & \\ & & 2 & 0 & 12 & & & \\ & & & 3 & 0 & 20 & & \\ & & & & \ddots & & \ddots & \\ & & & & & n-2 & 0 & n(n-2) \\ & & & & & & n-1 & 0 \\ & & & & & & & n \end{bmatrix}$$

where all the blank entries are 0. This is an upper triangular matrix, so its eigenvalues are the values on the diagonal, which are 0, 1, 2, …, n. In particular, n is an eigenvalue, so there is a polynomial $f(x)$ in this vector space with $Df(x) = nf(x)$.

Problem 3. In this problem, we will discuss the relevance of eigenvalues to oscillations of mechanical systems. If you hate physics, skip to the differential equation below.

There is a frictionless track with two masses resting on it, each of length m. There is a spring from the first mass to an anchor at the origin, and a spring between the two masses, each of which have rest length ℓ and spring constant k. So the masses would be at rest if they were at positions ℓ and 2ℓ.

Let the positions of the masses at time t be $\ell + x_1(t)$ and $2\ell + x_2(t)$. (So the x's are the displacement from the rest positions.) Then the masses obey the differential equations below.

$$\begin{array}{rcll} mx_1''(t) & = & -kx_1(t) & +k(x_2(t) - x_1(t)) \\ mx_2''(t) & = & & -k(x_2(t) - x_1(t)). \end{array}$$

(1) Find all solutions to these equations of the form $x_1(t) = a_1 \cos(\alpha t)$, $x_2(t) = a_2 \cos(\alpha t)$. (Hint: What does this have to do with eigenvalues?)
(2) Find a solution to these equations of the form $x_1 = a_1 \cos(\alpha t) + b_1 \cos(\beta t)$, $x_2 = a_2 \cos(\alpha t) + b_2 \cos(\beta t)$ with $x_1(0) = 0.1$ and $x_2 = -0.2$. (In other words, the masses start at positions $\ell + 0.1$ and $2\ell - 0.2$.

Solution (1) We want a solution of the form $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\alpha t)$. We have $\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = -\alpha^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\alpha t)$. We rewrite the right hand sides of the differential equations as

$$\begin{array}{rcl} -2kx_1(t) & + & kx_2(t) \\ kx_1(t) & - & kx_2(t) \end{array} = k \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

So, putting the parts together, we want to have

$$-m\alpha^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\alpha t) = k \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos(\alpha t).$$

Cancelling the $\cos(\alpha t)$ from each side and rearranging a little, we have

$$-\frac{m\alpha^2}{k} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

So we want $-\frac{m\alpha^2}{k}$ to be an eigenvalue of $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ and we want $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ to be an eignvector.

We thus compute the eigenvalues of $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$: The characteristic polynomial is

$$\det \begin{bmatrix} x+2 & -1 \\ -1 & x+1 \end{bmatrix} = (x+2)(x+1) - 1 = x^2 + 3x + 1.$$

The roots of this polynomial are $\frac{-3 \pm \sqrt{5}}{2}$, or about -3.618 and -0.382. So we have $\alpha = \sqrt{\frac{k(3 \pm \sqrt{5})}{2m}}$ The following isn't important but, incidentally, this can be simplified: It equals $\frac{\pm 1 + \sqrt{5}}{2} \sqrt{\frac{k}{m}}$, or roughly $1.618\sqrt{\frac{k}{m}}$ and $0.618\sqrt{\frac{k}{m}}$.

We also compute the eigenvectors. The $\frac{-3\pm\sqrt{5}}{2}$ eigenvector of $\left[\begin{smallmatrix} -2 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ is the kernel of

$$\begin{bmatrix} -2-\frac{-3\pm\sqrt{5}}{2} & 1 \\ 1 & -1-\frac{-3\pm\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1\mp\sqrt{5}}{2} & 1 \\ 1 & \frac{1\mp\sqrt{5}}{2} \end{bmatrix}$$

This kernel is spanned by $\left[\begin{smallmatrix} \frac{-1\pm\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right]$. In short, our solutions look like

$$a \left[\begin{smallmatrix} \frac{-1\pm\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right] \cos\left(\sqrt{\frac{k(3\pm\sqrt{5})}{2m}} t\right).$$

(2) Both sides of the differential equation are linear functions of $\left[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right]$, so any linear combination of solutions is another solution. So we look for solutions of the form

$$a \left[\begin{smallmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right] \cos\left(\sqrt{\frac{k(3+\sqrt{5})}{2m}} t\right) + b \left[\begin{smallmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right] \cos\left(\sqrt{\frac{k(3-\sqrt{5})}{2m}} t\right).$$

Plugging in $t = 0$, we have

$$a \left[\begin{smallmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right] + b \left[\begin{smallmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0.1 \\ -0.2 \end{smallmatrix}\right].$$

The solution to these equations is $a = b = -0.1$.