6.6.2 By assumption, every vector in V can be written in the form $\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k$ for $\vec{w}_i \in W_i$, so what we want to show is that there is only one such expression. There are many ways to do this, but here is a particularly nice way: Let $W_1 \oplus W_2 \oplus \cdots \oplus W_k$ be the abstract direct sum of the W_i. So $\dim(W_1 \oplus W_2 \oplus \cdots \oplus W_k) = \sum \dim W_i$. We have a map $W_1 \oplus W_2 \oplus \cdots \oplus W_k \longrightarrow V$ by $(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \mapsto \vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k$. Our assumption that $V = W_1 + W_2 + \cdots + W_k$ means that this map is surjective. But both vector spaces have the same dimension, so this shows it is also injective.

6.6.4 False. We'll take our field to be $\mathbb{R}$. The map $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a projection onto $\mathbb{R}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and the map $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is a projection onto $\mathbb{R}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. But $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is not a projection. Indeed, we check that $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}^2 = \begin{bmatrix} 5/2 & 1 \\ 1 & 1/2 \end{bmatrix} \neq \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

6.6.6 True. Choosing a basis where our operator is diagonal, we can assume E is of the form

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}.$$

Then $E^2 = E$ as desired.

6.7.1 Suppose first that $ETE = TE$ and let $\vec{y} \in \operatorname{Im}(E)$, meaning that $\vec{y} = E\vec{x}$ for some $\vec{x}$. Then $T\vec{y} = TE\vec{v} = ETE\vec{x} = E(TE\vec{x})$, so $T\vec{y} \in \operatorname{Im}(E)$ as desired.

Conversely, suppose that T maps $\operatorname{Im}(E)$ to itself. For any vector $\vec{x}$, we then have $TE\vec{x} \in \operatorname{Im}(E)$. But E maps every vector in $\operatorname{Im}(E)$ to itself, so we deduce that $E(TE\vec{x}) = TE\vec{x}$, as desired.

Now, suppose that $ET = TE$. This implies that $ETE = TE^2 = TE$, so we conclude that T maps $\operatorname{Im}(E)$ to itself as desired. We must check that T also maps $\operatorname{Ker}(E)$ to itself. Indeed, let $E\vec{z} = \vec{0}$. Then $E(T\vec{z}) = TE\vec{z} = T\vec{0} = \vec{0}$,. showing that $T\vec{z}$ is in $\operatorname{Ker}(E)$ as desired.

Finally, suppose that T maps $\operatorname{Im}(E)$ and $\operatorname{Ker}(E)$ to themselves. Since $V = \operatorname{Im}(E) \oplus \operatorname{Ker}(E)$, every vector in V can be written as $\vec{y} + \vec{z}$ for $\vec{y} \in \operatorname{Im}(E)$ and $\vec{z} \in \operatorname{Ker}(E)$. By the definition of $\operatorname{Im}(E)$, we rewrite this as $\vec{v} = E\vec{x} + \vec{z}$. We must check that $ET(E\vec{x} + \vec{z}) = TE(E\vec{x} + \vec{z})$. On the left hand side, we have $ET(E\vec{x} + \vec{z}) = ETE\vec{x} + ET\vec{z}$. By our assumption on T, the vector $TE\vec{x}$ is in $\operatorname{Im}(E)$, so $ETE\vec{x} = TE\vec{x}$. Also by our assumption on T, the vector $T\vec{z}$ is in $\operatorname{Ker}(E)$, so $ET\vec{z} = \vec{0}$. We conclude that the left hand side is $TE\vec{x}$. On the right hand side, we have $TE(E\vec{x} + \vec{z}) = TE^2\vec{x} + \vec{0} = TE\vec{x}$. So both sides are $TE\vec{x}$ and we are done.

6.7.2 (a) We check that $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right] = 2\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

(b) Any space complementary to $\mathbb{R}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ must be of the form $\mathbb{R}\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ for some x. But then $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2x+1 \\ 2 \end{smallmatrix}\right]$. If $\mathbb{R}\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ were invariant, then the vector $\left[\begin{smallmatrix} 2x+1 \\ 2 \end{smallmatrix}\right]$ would have to be in $\mathbb{R}\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$. But $\det\left[\begin{smallmatrix} 2x+1 & x \\ 2 & 1 \end{smallmatrix}\right] = 1 \neq 0$, so there is no x for which $\left[\begin{smallmatrix} 2x+1 \\ 2 \end{smallmatrix}\right]$ is a multiple of $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$.

6.8.1 The characteristic polynomial of T is $x^3 - 2x^2 + x - 2 = (x-2)(x^2+1)$. We take $p_1(x) = x - 2$ and $p_2(x) = x^2 + 1$. We compute bases for $p_1(T)$ and $p_2(T)$. We have

$$p_1(T) = T - 2\mathrm{Id}_3 = \begin{bmatrix} 4 & -3 & -2 \\ 4 & -3 & -2 \\ 10 & -5 & -5 \end{bmatrix} = \mathbb{R}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$p_2(T) = T^2 + \mathrm{Id}_3 = \begin{bmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{bmatrix} = \mathbb{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

6.8.9 One example is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Both of these matrices have minimal polynomial x^2.