Problem Set 1 – Due Tuesday, September 10

Please see the course website for homework policy.

I aimed high on this problem set; some of the problems are quite hard. Please try them all, but you may not be able to get them all.

Problem 1 Let G be a connected graph with equally many vertices and edges. Show that G has exactly one cycle.

Problem 2 Let G be a directed graph on a finite vertex set V.

(a) Suppose that every vertex of G has out-degree 1. Show that G has a directed cycle.

(b) Suppose that $v \in V$ is a vertex of out-degree 0 and every vertex other than v has out-degree 1. Show that the following are equivalent:

(i) G, considered as an undirected graph, is connected

(ii) G, considered as an undirected graph, is a tree

(iii) G, considered as an undirected graph, has no cycles

(iv) G, considered as a directed graph, has no directed cycles

Problem 3 Let T be a tree all of whose vertices have degree either 1 or 3. Such a tree is called *trivalent* and often occur in evolutionary biology, describing how various species have branched apart from each other.

(a) If T has n leaves, show that it has $n - 2$ vertices of degree 3.

(b) Let T be a trivalent tree with $n \geq 4$. Show that there is some internal vertex which is adjacent to two leaves.

Such a vertex is sometimes called a *cherry*, and many algorithms for phylogenetic reconstruction begin by trying to find the cherries.

Problem 4 Let T be a tree.

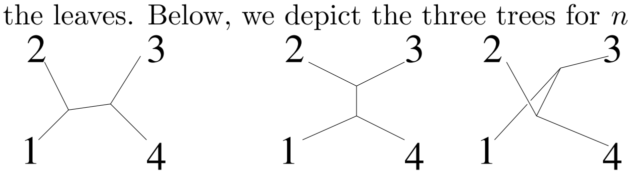
(a) Show that it is possible to color the vertices of T black and white so that neighboring vertices have opposite colors.

Let b and w be the numbers of black and white vertices.

(b) If $b \geq w$, show that T has a black leaf.

(c) Let ℓ be the number of leaves of T. Show that $|b - w| < \ell$ (unless T is a single vertex).

Problem 5 Consider trivalent trees (defined in Problem 2) whose leaves are numbered 1, 2, …, n. We consider two such trees T and T' to be the same if there is an isomorphism $T \cong T'$ preserving the labels of the leaves. Below, we depict the three trees for $n = 4$.



(a) How many such trees are there for $n = 5$, 6 and 7? (Don't write them out!)

(b) Conjecture a formula for the number of such trees with n leaves.

(c) Prove your guess.

Problem 6 Consider a $(2n + 1) \times (2n + 1)$ checkerboard. Place $2n^2 + 2n$ dominos on the checker board, leaving one corner uncovered. Show that it is possible to slide the dominos in order to move the hole to any position whose x and y coordinates have the same parity as the initial corner. (These positions are marked with red dots in the image.)