Problem Set 2 – Due Tuesday, September 17

Please see the course website for homework policy.

Problem 1 Let $K_{p,q}$ be the graph with vertices $v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_q$ and with an edge from v_i to w_j for each i and j. (So there are pq edges in all.) Let $L_{p,q}$ be the Laplacian matrix of $K_{p,q}$.

(a) Compute the characteristic polynomial of $L_{p,q}$ for enough values of (p,q) to make a guess as to the general answer.

(b) Prove your guess.

(c) How many spanning trees does $K_{p,q}$ have? (feel free to answer this on the basis of your guess, even if you haven't proven your guess.)

Problem 2 Let G_d be the graph which has $2d$ vertices, $u_1, \ldots, u_d, v_1, \ldots, v_d$ and where every vertex is connected to every other vertex *except* that there is no edge from u_i to v_i.

Let S be the $2d \times 2d$ matrix with a 1 in positions $(i, i+d)$ and $(i+d, i)$, and zeroes everywhere else. Let J be the $2d \times 2d$ matrix whose every entry is a 1.

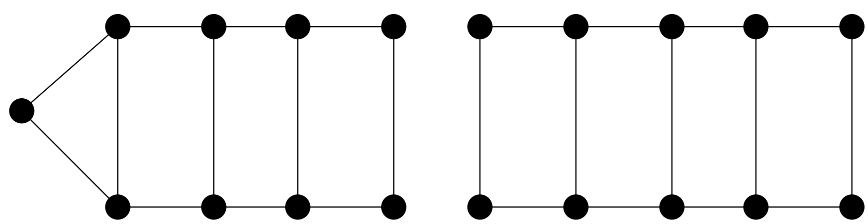
(a) Express the Laplacian matrix of G_d in terms of S, J and Id.

(b) Check that $S^2 = \mathrm{Id}$, $SJ = JS = J$ and $J^2 = 2dJ$. What are the possible eigenvalues of S and J?

(c) Show that S and J can be simultaneously diagonalized. Describe the corresponding eigenspaces and eigenvalues.

(d) How many spanning trees does G_d have?

Problem 3 Let G_n and H_n be the graphs defined by example below:



G_4 H_4

So G_n has $n-1$ rectangles and one triangle and H_n has n rectangles. Let $\tau(G)$ denote the number of spanning trees of the graph G. Prove that

$$\begin{aligned}\tau(G_n) &= \tau(G_{n-1}) + 2\tau(H_{n-1})\\ \tau(H_n) &= \tau(G_{n-1}) + 3\tau(H_{n-1})\end{aligned}$$

(It is easy to solve these recursions and obtain closed formulas for $\tau(G_n)$ and $\tau(H_n)$, although you are not required to do this.)

Problem 4 Let T_n be the number of trees on vertex set $\{1, 2, \ldots, n\}$. (We will prove in class that $T_n = n^{n-2}$, but it is easiest to do this problem without using that fact.)

Prove that

$$2(n-1)T_n = \sum_{k=1}^{n-1} \binom{n}{k} k T_k (n-k) T_{n-k}.$$