Problem Set 3 – Due Thursday, September 26

Please see the course website for homework policy.

Problem 1 Let G be a graph with n vertices and e edges, so that every edge lies in m spanning trees. Consider the graph as made of identical resistors with resistance R. What is the effective resistance between two adjacent vertices?

Problem 2 Let n be a positive integer. Construct a binary string in a greedy manner as follows: Start with $n-1$ copies of 0. Add bits to the end as follows: Add 1 unless this would cause the same n-bit substring to occur twice. If we are forbidden to add 1, then add 0 if this will not cause an n-bit substring to reoccur. If both 1 and 0 would cause an n-bit substring to reoccur, then stop. For $n = 3$, this produces

$$0011101000.$$

In this problem, we will show that this procedure ends with $n-1$ zeroes and that gluing those zeroes to the initial zeroes produces a de Bruijn cycle.

Let D_n be the directed graph whose vertices are binary strings of length $n-1$ and where there is an edge from $b_1 b_2 \cdots b_{n-1}$ to $b_2 b_3 \cdots b_{n-1} b_n$. (This is the graph we used in class to prove de Bruijn sequences exist.) Let T be the subgraph consisting of the edges $b_1 b_2 \cdots b_{n-1} \longrightarrow b_2 \cdots b_{n-1} 0$ for every binary string $b_1 b_2 \cdots b_{n-1}$ other than $00 \cdots 0$.

(a) For $n = 3$, draw the graph D_3 and the subgraph T. You should see that T is a rooted subtree of D_3; construct the corresponding de Bruijn cycle.

(b) Prove in general that T is a rooted subtree of D_n.

(c) Prove that the Eulerian walk of D_n corresponding to T (by the BEST algorithm) gives the de Bruijn sequence constructed by the greedy procedure above.

Problem 3 Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$ arranged around a circle. G has $4n$ edges: For each vertex, there are two edges to its clockwise neighbor and two to its counter-clockwise neighbor.

(a) Find the number of rooted spanning trees of G. (Just thinking about the problem is probably easier than using the matrix-tree theorem.)

(b) Find the number of Eulerian walks in G.

Problem 4 Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be subsets of the plane $\mathbb{R}^2$. For any subset I of $\{1, 2, \ldots, n\}$, let $Y_I = \left(\bigcap_{i \in I} X_i\right) \cap \left(\bigcap_{j \notin I} X_j^C\right)$, where X_j^C is $\mathbb{R}^2 \setminus X_j$. We will define X_1, $\ldots$, X_n to be a generalized Venn diagram if all the X_i's are convex and all the Y_I's are nonempty.

Show that there exist generalized Veen diagrams for any n.

Hint: The image above shows one of the sets X_i for $n = 3$; the dashed lines are an aid to visualization.