Problem Set 5 – due October 22nd

Note the later due date, as a result of Fall Break. Please see the course website for homework policy.

Problem 1 Let G be a finite graph and let A be its adjacency matrix. Let d be the largest degree of any vertex of G.

(a) Show that the eigenvalues of A lie in the interval $[-d, d]$.

(b) Give a simple criterion for when d is an eigenvalue of A.

(c) Give a simple criterion for when $-d$ is an eigenvalue of A.

Problem 2 Let G be the Petersen graph and A it's adjacency matrix. We showed in class that A has eigenvalues 3, 1, 1, 1, 1, 1, -2, -2, -2, -2.

Let v be a vertex of G. Give a formula for the number of walks G of length k, starting and ending at v. Hint: For $k = 1$, 2 and 3, you should get 0, 3 and 0.

Problem 3 Let G be a finite graph with adjacency matrix A and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Show that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ is a nonnegative integer and give a combinatorial description of it.

(b) Show that $\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_n^3$ is a nonnegative integer and give a combinatorial description of it.

(c) Suppose that G has n vertices and every vertex has degree d. Show that

$$\lambda_1^4 + \lambda_2^4 + \cdots + \lambda_n^4 - (nd + nd(d-1) + nd(d-1))$$

is a nonnegative integer and give a combinatorial description of it. (The expression in parentheses is deliberately not simplified as a hint.)

Problem 4(a) Let G be a finite graph. Let w_k be the number of closed walks in G of length k and let p be a prime number. Show that w_p is divisible by p.

(b) Give an example of a finite graph for which w_4 is not divisible by 4.

(c) Let A be an $n \times n$ integer matrix and let p be a prime. Show that

$$\mathrm{Tr}(A^p) \equiv \mathrm{Tr}(A) \bmod p.$$

You will want Fermat's Little Theorem, which says that $a^p \equiv a \bmod p$ for any integer a. (Hint: Thinking about walks in graphs helps.)

(d) Find an integer matrix A and a prime p such that $A^p \not\equiv A \bmod p$.

Problem 5 Let G be a finite graph with at least two vertices such that, for any two distinct vertices u and v, there are exactly two paths of length 2 from u to v.

(a) Show that all vertices of G have the same degree.

Let d be this common degree and let n be the number of vertices of A.

(b) Give a formula for n in terms of d.

(c) By analyzing the spectrum of A, show that there are only two possible values of d. (If you want to check your answer, the following is in ROT13: Ubj znal oevqrf qvq Qenphyn unir? Ubj nobhg Urael gur Rvtugu?)

(d) Construct graphs achieving each of the values of d you found in the previous part.