Problem Set 6 – due October 29nd

Please see the course website for homework policy.

Problem 1 Let G be a graph and let $L(G)$ be the Laplacian matrix of G. Let $\lambda_2(G)$ be the second smallest eigenvalue of $L(G)$. (The smallest eigenvalue is 0, corresponding to the all 1 vector.) Hint (ROT13): Guvf ceboyrz vf nobhg gur Enyrvtu dhbgvrag.

(a) Let G' be a graph formed by adding an extra edge to G. Show that $\lambda_2(G') \geq \lambda_2(G)$.

(b) **I am no longer sure this is true, though I can't break it. Please let me know if you find a proof.** Let v be a vertex of G, with neighbors $u_1, u_2, \ldots, u_k$. Choose i with $1 \leq i \leq k$ and make a new graph G' as follows: replace v by two vertices x and y, where x is joined to $u_1, u_2, \ldots, u_i$ and y is joined to $u_{i+1}, u_{i+2}, \ldots, u_k$. Show that $\lambda_2(G') \leq \lambda_2(G)$.

Problem 2 Let H be a bipartite graph and let $A(H)$ be its adjacency matrix. Show that, if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H.

Problem 3 Let G be a d-regular graph on n vertices. Let $A(G)$ be the adjacency matrix of G, with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) Show that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = dn$.

(b) Show that $\max(|\lambda_2|, |\lambda_n|) \geq \sqrt{\frac{dn-d^2}{n-1}}$.

(c) Fix d and fix $\epsilon > 0$. Show that, for n sufficiently large, $\max(|\lambda_2|, |\lambda_n|)$ is $> \sqrt{d} - \epsilon$.

Problem 4.(a) Let G be the n-cycle and let $L(G)$ be the Laplacian matrix of G. Show that the eigenvalues of $L(G)$ are $2 - 2\cos(2\pi j/n)$, for $0 \leq j < n$.

(b) Let H be the path of length n and let $L(H)$ be the Laplacian matrix of H. Let G be the $2n$-cycle with Laplacian matrix $L(G)$. Show that every eigenvalue of $L(H)$ is also an eigenvalue of $L(G)$. More specifically, show that the eigenvalues of H are $2 - 2\cos(2\pi j/(2n))$, for $0 \leq j < n$.

(c) Let S be the tree with $n+1$ vertices, one of which borders all the others. Find the eigenvalues of $L(S)$.