Problem Set 7 – due November 5th

Please see the course website for homework policy.

Problem 1 Fix a positive integer d; in this problem we will study d-regular graphs on n vertices. Let $A(G)$ be the adjacency matrix and λ_i its eigenvalues. On the previous problem set, we proved the following: For any $\epsilon > 0$, there is an N such that, if $n > N$, then $\max(|\lambda_2|, |\lambda_n|) > \sqrt{d} - \epsilon$. In this problem, we will improve $\sqrt{d}$ to $\sqrt{2d-1}$. Fix $c > 0$.

(a) Let $B = A^2 - (d - (1+c)/2)$. Show that $Tr(B^2) \geq n(d^2 - d + (1+c)^2/4)$. When does equality occur?

Let $g = \sqrt{2d-1-c}$.

(b) Suppose that all the eigenvalues of A, other than d, are in $(-g, g)$. Show that $Tr(B^2) \leq (d^2 - d + (1+c)/2)^2 + (n-1)(d - (1+c)/2)^2$.

(c) Show that the above inequalities imply a finite upper bound for n (dependent on c and d).

Problem 2 Let G be a graph. We define a double of G to be a graph DG as follows: Each vertex v in G gives two distinct vertices v_1 and v_2 in DG. If there is an edge (v, w) in G, then *either* there are edges (v_1, w_1) and (v_2, w_2), or else there are edges (v_1, w_2) and (v_2, w_1) (but not both). There are no other edges in DG.

(a) Show that every eigenvalue of $A(G)$ is an eigenvalue of $A(DG)$.

(b) Define a matrix B as follows: there is a row and a column of B for every vertex of G. For two vertices v and w of G, we have $B_{vw} = 0$ if there is no edge (v, w) in G; we have $B_{vw} = 1$ if edges (v_1, w_1) and (v_2, w_2) occur in DG; and we have $B_{vw} = -1$ if edges (v_1, w_2) and (v_2, w_1) occur in DG. Show that every eigenvalue of B is an eigenvalue of $A(DG)$.

Problem 3 Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \cdots \lambda_n$. Let B be the upper left $m \times m$ submatrix, with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$.

(a) Show that $\mu_1 \leq \lambda_1$.

(b) Let $\vec{v}_i$ be the eigenvectors of A. Show that there is some vector $\vec{u}$ which is supported in the first m-coordinates and is in the span of $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_{n-m}$, $\vec{v}_{1+n-m}$. Use this fact to show that $\mu_1 \geq \lambda_{1+n-m}$.

(c) If we repeat these arguments for μ_m, what bounds do we get?

Let $k \leq m$. In the next two parts, we will find bounds for μ_k in terms of the λ_i.

(d) Write C for the upper left $(m-k+1) \times (m-k+1)$ submatrix and γ_i for its eigenvalues. Show that we can change bases, without changing the eigenvalues of A and B, so that $\mu_k = \gamma_1$. Deduce that $\mu_k \geq \lambda_{k+n-m}$.

(e) Write D for the upper left $k \times k$ submatrix and δ_i for its eigenvalues. Show that we can change bases, without changing the eigenvalues of A and B, so that $\mu_k = \delta_k$. Deduce that $\mu_k \leq \lambda_k$.

(f) The above results give some very crude restrictions on subgraphs of Ramanujan graphs. For example, suppose that G is a 10 regular graph with all eigenvalues of the adjacency matrix other than λ_1 in $[-6, 6]$. Show that G does not contain two seven element subsets X and Y so that there are edges joining every $x \in X$ to every $y \in Y$, and so that there are no edges within X or within Y.