Problem Set 7 – due November 19th

You have two weeks for this problem set, since there will be no class on November 7 and since November 5 will be a wrap up of the material on spectral graph theory. Problems 1 and 2 are designed to use only cleverness without theory.

Please see the course website for homework policy.

Problem 1 Consider a $(2n+1)\times(2n+1)$ checker board, with the corners colored black. Suppose that we remove any black square from the board, living $4n^2+4n$ squares behind. Show that the remaining squares can be tiled with dominos. (A direct proof is probably easier than appealing to Hall's marriage theorem.)

Problem 2 Let G be a graph. By definition, a **perfect matching** of G is a collection M of edges of G so that every vertex of G lies on exactly one edge of M. Let $(v_1, v_2, \ldots, v_{2k})$ be a cycle of G of even length. If M is a perfect matching of G which contains the edges (v_1, v_2), (v_3, v_4), …, (v_{2k-1}, v_{2k}), then define the **twist** of M along $(v_1, v_2, \ldots, v_{2k})$ to be the perfect matching which deletes the edges (v_1, v_2), (v_3, v_4), …, (v_{2k-1}, v_{2k}) from M and replaces them by the edges (v_2, v_3), (v_4, v_5), …, (v_{2k-2}, v_{2k-1}), (v_{2k}, v_1).

(a) If M and M' are two perfect matchings of G, show that we can change M to M' by a sequence of twists along various cycles of G.

An **induced cycle** of G is a cycle $(v_1, v_2, \ldots, v_m)$ so that G has no edges among the vertices v_i other than the m edges of the cycle.

(b) Prove or disprove: If M and M' are two perfect matchings of G, then we can change M to M' by a sequence of twists along various induced cycles of G.

(c) Prove or disprove: If G is bipartite and M and M' are two perfect matchings of G, then we can change M to M' by a sequence of twists along various induced cycles of G.

Let G be a graph drawn in the plane without crossing itself. We'll say that a cycle of G bounds a face if there are no edges of G inside the part of $\mathbb{R}^2$ enclosed by that cycle.

(d) (Harder) Prove or disprove: Let G be a connected bipartite planar graph and let M and M' be two perfect matchings. Then we can change M to M' by a sequence of twists along cycles which bound faces.

Problem 3 We deal a deck of cards into 13 piles of 4. Show that it is possible to pick up one card from each pile and have precisely one card of each rank: One ace, one deuce and so forth, up to one king.

Problem 4 Let A be a square $n \times n$ matrix. I'll say that N is a **magic square** if there is some constant N such that every row and every column[1] of A sums to N. A **permutation matrix** is a $(0,1)$ matrix where every row and every column contains exactly one 1; so a permutation matrix is a magic square with $N = 1$.

(a) Show that, if A is a magic square with nonnegative integer entries, then A is a sum of N permutation matrices. (Hint in ROT13: Svefg gel gb fubj gung gurer vf n fvatyr crezhgngvba zngevk P fhpu gung $A - P$ fgvyy unf abaartngvir vagrtre ragevrf.)

(b) Show that, if A is a magic square with nonnegative real entries, then A is a positive linear combination of permutation matrices.

(c) (Harder) Let A be an $n \times n$ magic square with nonnegative real entries. Show that there are $(n-1)^2+1$ permutation matrices, P_1, P_2, …, $P_{(n-1)^2+1}$ so that $A = \sum c_i P_i$ with the c_i nonnegative real numbers.

[1] We don't impose this condition on the diagonals.