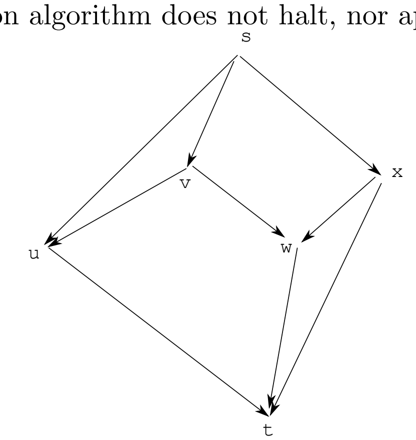
Problem Set 9 – due November 26th

Please see the course website for homework policy.

Problem 1.(a) Let G be a directed graph with source s, sink t and edge capacities $c(e)$. Suppose that all the capacities $c(e)$ are integers. Show that there is an optimal flow through G with integer amounts of flow through every edge.

1.(b) Let H be a bipartite graph. Construct a directed graph G, source s, sink t and some edge capacities $c(e)$, such that the maximal flow from s to t is equal to the maximum cardinality of any matching of G.

Problem 2 The aim of this problem is to show an example of a graph with irrational edge weights where the Ford-Fulkerson algorithm does not halt, nor approach the correct limit.



Our graph is shown in the image above. The capacities are as follows:

$$c(v,u) = c(v,w) = 1, \quad c(x,w) = \frac{\sqrt{5}-1}{2} \approx 0.618, \quad \text{all other edges have } c = 10.$$

2.(a) Compute the maximum possible flow through G. Give an example of a cut whose capacity equals this flow.

Define the paths $p_1 = (s \to x \to w \to v \to u \to t)$, $p_2 = (s \to v \to w \to x \to t)$ and $p_3 = (s \to u \to v \to w \to t)$.

2.(b) Start with the flow which is 1 on $s \to v \to w \to t$. Successively increment it, as much as possible,

Take the empty flow and successively increment it, as much as possible, along p_1, p_2, p_1, p_3, $\ldots$, with the patten repeating with period 4. Compute that first 4 flows you produce in this way and their residual graphs.

2.(c) Show that, no matter how many times you go through the procedure in 2.(b), you'll never get to even half the total capacity of the network.

Problem 3 Let G be a graph whose edges are assigned lengths. Let t be the length of the shortest spanning tree of G. Let s be the length of the shortest path which visits every vertex.

3.(a) Show that $t \leq s$.

3.(b) Show that $s \leq 2t$.

Computing s is known as the traveling salesman problem, and is famously difficult. Nonetheless, this argument shows that it is easy to get a crude approximation to s.

3.(c) Give an example of a graph G, with lengths assigned to edges with the following property: For any spanning tree T of G, there is some pair of vertices so that the distance from u to v in T is ≥ 100 times larger than the distance between u and v in G.

So this trick does not help for diameters.