

SOLUTION SET 1

Problem 1 Let G be a connected graph with equally many vertices and edges. Show that G has exactly one cycle.

Let G have n vertices and n edges. Since G is a connected graph, it has a spanning tree T with n vertices and $n - 1$ edges. Let e be the edge not in T , with its endpoints u and v . There is a unique path γ between u and v in T (since T is a tree). The union of e and γ is a cycle.

Suppose that there is some other cycle δ . If δ does not contain e , then it is contained in T , contradicting that T has no cycles. If δ does contain e , write it as the union of e and a path ϵ in T . Then ϵ is a path from u to v . But γ is the only path from u to v in T .

Problem 2 Let G be a directed graph on a finite vertex set V .

(a) Suppose that every vertex of G has out-degree 1. Show that G has a directed cycle.

Start at a vertex w_0 . Let w_1 be a vertex with an edge $w_0 \rightarrow w_1$. Let w_2 be a vertex with an edge $w_1 \rightarrow w_2$. Continue in this manner, letting w_{i+1} be a vertex with an edge $w_i \rightarrow w_{i+1}$. Since V is finite, sooner or later the same vertex will occur twice in this sequence. So there is a path $w_i \rightarrow w_{i+1} \rightarrow \dots \rightarrow w_j$ for some $i < j$. This is a directed cycle.

(b) Suppose that $v \in V$ is a vertex of out-degree 0 and every vertex other than v has out-degree 1. Show that the following are equivalent:

- (i) G , considered as an undirected graph, is connected
- (ii) G , considered as an undirected graph, is a tree
- (iii) G , considered as an undirected graph, has no cycles
- (iv) G , considered as a directed graph, has no directed cycles

Let V have n vertices. Every edge points out of one vertex, so the number of edges is $\sum_{v \in V} \text{outdegree}(v) = 1 + 1 + \dots + 1 + 0 = n - 1$. Therefore, (i), (ii) and (iii) are equivalent. Also, it is obvious that (iii) implies (iv). It remains to show that (iv) implies one of (i), (ii) and (iii); we will show that (iv) implies (i).

Assume (iv). Consider any vertex w_0 in V and construct w_1, w_2, \dots as above for as long as you can. Since G has no directed cycles, the process must stop at some w_i , which must be v . Then we have a directed path $w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_i = v$. We have shown that there is a path from an arbitrary vertex to v , so G is connected.

In fact, as came up in class during the proof of the matrix-tree theorem, these conditions are equivalent to G being a rooted tree.

Problem 3 Let T be a tree all of whose vertices have degree either 1 or 3. Such a tree is called trivalent and often occur in evolutionary biology, describing how various species have branched apart from each other.

(a) If T has n leaves, show that it has $n - 2$ vertices of degree 3.

Let m be the number of vertices of degree 3. So there are $m + n$ vertices in general, and thus $m + n - 1$ edges. The sum of the degrees of the vertices is twice the number of edges, so $n + 3m = 2((m + n) - 1)$. Rearranging, $m = n - 2$.

(b) Let T be a trivalent tree with $n \geq 4$. Show that there is some internal vertex which is adjacent to two leaves.

There are many ways to prove this, here is one. Let U be the subgraph on T on the internal vertices. Then U has no cycles, since T has no cycles, and U is connected, since any two internal vertices of T can be joined by a path which doesn't use a leaf. Finally, since $n \geq 3$, we see that U has at least one vertex and is, in particular, nonempty. Thus, U is a tree. Moreover, since $n \geq 4$,

U has at least two vertices and therefore has a leaf ℓ . The vertex ℓ must have bordered two leaves of T .

Problem 4 Let T be a tree.

(a) Show that it is possible to color the vertices of T black and white so that neighboring vertices have opposite colors.

We work by induction on the number of vertices of T . If T is a single vertex, then it can be colored (either black or white). If T has more than one vertex, then it has a leaf ℓ ; let t be the vertex neighboring ℓ . The graph formed by deleting ℓ from T is again a tree so, inductively, it can be bipartitely colored. If t is colored black, then color ℓ white, and vice versa.

Let b and w be the numbers of black and white vertices.

(b) If $b \geq w$, show that T has a black leaf.

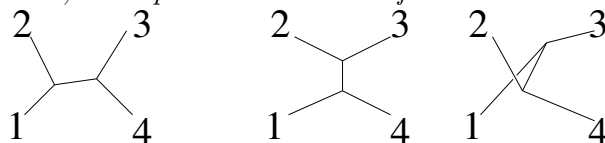
We prove the contrapositive: If all the leaves of T are white, then $w > b$. Suppose, therefore, that all the leaves of T are white. Let ℓ_1 be the number of leaves of T . Let T_2 be the tree that results by deleting all the leaves of T , so all the leaves of T_2 are black. Let ℓ_2 be the number of leaves of T_2 and let T_3 be the tree that results from deleting all the leaves of T_2 . By induction, T_3 has more white vertices than black, so we are done if we show that $\ell_1 \geq \ell_2$. Indeed, every vertex of T_2 borders at least one leaf of T , and every leaf of T borders precisely one leaf of T_2 , so there is a surjection from leaves of T to leaves of T_2 . (This is only one of many ways to prove this.)

(c) Let ℓ be the number of leaves of T . Show that $|b - w| < \ell$ (unless T is a single vertex).

Let ℓ_b be the number of black leaves and ℓ_w the number of white leaves. If we delete all the black leaves then the resulting tree only has white leaves so, by the contrapositive of part (b), $w - (b - \ell_b) > 0$. Similarly, $b - (w - \ell_w) > 0$. So $\ell_b > b - w > -\ell_w$ and we see that $|b - w| < \max(\ell_b, \ell_w) \leq \ell$.

Attribution: This is a lemma from my first paper, *Every tree is 3-equitable*, with Zsuzsanna Szaniszló, *Discrete Math* (2000). I list it to point out that lemmas in research papers are often no harder than problems on problem sets.

Problem 5 Consider trivalent trees (defined in Problem 2) whose leaves are numbered $1, 2, \dots, n$. We consider two such trees T and T' to be the same if there is an isomorphism $T \cong T'$ preserving the labels of the leaves. Below, we depict the three trees for $n = 4$.



(a) How many such trees are there for $n = 5, 6$ and 7 ? (Don't write them out!)

The numbers are 15, 105 and 945. Probably the easiest way to do this count is to first list the trees up to isomorphism: There are 1, 2 and 2 such trees respectively, and they contribute $15, 90 + 15$ and $630 + 315$ trees respectively.

(b) Conjecture a formula for the number of such trees with n leaves.

We are looking at the sequence 1, 3, 15, 105, 945. These numbers all have very small prime factors, so some sort of multiplicative formula seems reasonable. In fact, each number divides the previous number: We have $3/1 = 3, 15/3 = 5, 105/15 = 7$ and $945/105 = 9$. This suggests that the number of trivalent trees with n leaves is $1 \times 3 \times 5 \times 7 \times \dots \times (2n - 5)$.

Another resource you could (and may) use is Sloane's online encyclopedia of integer sequences: <http://www.oeis.org>. There is one sequence in the encyclopedia containing 1, 3, 15, 105, 945; you can see it at <http://oeis.org/A001147>.

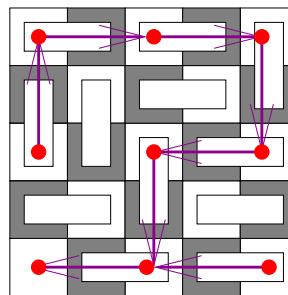
(c) Prove your guess.

Let T_n be the set of trivalent trees with leaves labeled by n . We will create a map $T_n \rightarrow T_{n-1}$ so that each tree in T_{n-1} is the image of $2n - 5$ trees; the formula will then follow by induction.

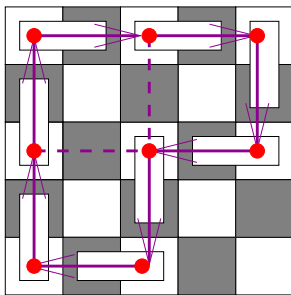
Here is the map. Let T be a trivalent tree with leaves $\{1, 2, \dots, n\}$ and $n \geq 1$. Let u be the vertex of T bordering n . Assuming $n \geq 3$, u will be an internal vertex; let its other neighbors be x and y . Delete the leaf n and the vertex u , then join x and y by a new edge. Call the resulting tree T' , so T' is a trivalent tree with $n - 1$ leaves.

Given a tree T' , how many trees can it come from? For each edge (x, y) of T' , we can insert a vertex u in the middle of it and connect it to a new leaf labeled n ; this produces a tree T which maps to T' . So the number of trees mapping to T' is the number of edges of T' . Since T' has $n - 1$ leaves and $(n - 1) - 2$ internal vertices, it has $(n - 1) + (n - 3) - 1 = 2n - 5$ edges. So, for each T' in T_{n-1} , there are $2n - 5$ trees T mapping to T' , as desired.

Problem 6 Consider a $(2n + 1) \times (2n + 1)$ checkerboard. Place $2n^2 + 2n$ dominos on the checker board, leaving one corner uncovered. Show that it is possible to slide the dominos in order to move the hole to any position whose x and y coordinates have the same parity as the initial corner.



We begin by building a directed graph G . The vertices of this graph are the positions whose x and y coordinates have the same parity as the initial corner. If (x, y) is a vertex of G other than the corner, then there is a domino covering (x, y) . Let the other square of that domino be $(x, y) + (\delta, \epsilon)$, where (δ, ϵ) is one of $(1, 0)$, $(0, 1)$, $(-1, 0)$ or $(0, -1)$. Then, in the graph G , we draw an edge from (x, y) to $(x, y) + 2(\delta, \epsilon)$. So every vertex of G has outdegree 1 except the initial hole, which has outdegree 0. (In other words, this is a graph of the sort you studied in problem 2(b).) In the above figure, the graph G is the bold purple lines. (The purpose of the dashed lines will be explained later.)



The crucial claim is that this graph has no directed cycles. The figure to the left shows an arrangement of dominoes which would give an oriented cycle. However, notice that this arrangement cannot be completed to a full tiling because the region inside the cycle has 5 squares. We will show that this is the general situation. **Claim:** Any oriented cycle encloses an odd number of squares, and therefore cannot be complete to a tiling of the full board.

So, suppose that we have an oriented cycle γ ; we want to show that γ encircles an odd number of squares of the original checker board. One can prove this in a tedious way by induction on the length of γ , but let's take a slick route. Take the area inside γ and cut it up into 2×2 squares, as shown by the dashed lines. Let H be the graph made up of γ and the dashed lines. Let F be the number of 2×2 squares in H ; let E be the number of edges of H and let V be the number of vertices of H . (In the example, $F = 3$, $E = 10$ and $V = 8$.) Euler's relation gives $V - E + F = 1$. (We will discuss Euler's relation when we cover planar graphs.) Let the length of the cycle γ be n (so, in this case, $n = 8$).

The checkerboard squares inside γ whose coordinates are of the same parity as the hole correspond to the vertices of H not on γ ; so there are $V - n$ of them. The checkerboard squares inside γ whose coordinates are both of the opposite parity as the hole correspond to the 2×2 faces of H ; there are F of them. And the checkerboard squares inside γ which have one coordinate of one parity and one of the other correspond to the dashed edges of H , which there are $E - n$ of. So, in total, the number of squares inside γ is

$$(V - n) + F + (E - n) = V + F + E - 2n \equiv V - E + F = 1 \pmod{2}.$$

This completes the proof of the claim.

So, for any tiling of the complete checker board, the graph G cannot have an oriented cycle. Start at any vertex v_0 of G and follow the directed edges. Since there is no directed cycle, you must eventually reach the hole. So there is a directed path $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$ where v_k is the hole. Slide the domino on v_{k-1} to v_k ; slide the domino on v_{k-2} to v_{k-1} and so forth, until you have moved the hole to v_0 .

Attribution: This problem appeared on the 1997 All-Russian Mathematical Olympiad, credited to A. Shapovalov. It is also a special case of a relation between matchings and spanning trees in planar graphs which we may discuss later.