Solution Set 2

Problem 1 *Let $K_{p,q}$ be the graph with vertices $v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_q$ and with an edge from v_i to w_j for each i and j. (So there are pq edges in all.) Let $L_{p,q}$ be the Laplacian matrix of $K_{p,q}$.*

(a) Compute the characteristic polynomial of $L_{p,q}$ for enough values of (p,q) to make a guess as to the general answer.

With the help of a computer algebra system, you should quickly come up with the guess

$$x(x+q)^{p-1}(x+p)^{q-1}(x+p+q).$$

(b) Prove your guess.

We need to find enough eigenvectors. The all ones vector accounts for the 0 eigenvalue.

For any v_i and v_j, let $\vec{x}(i,j)$ be the vector which is 1 on v_i and -1 on v_j. Then $L\vec{x}(i,j) = q\vec{x}(i,j)$. The $\vec{x}(i,j)$ span a $(p-1)$-dimensional eigenspace with eigenvalue q. Similarly, for any w_i and w_j, let $\vec{y}(i,j)$ be the vector which is 1 on w_i and -1 on w_j; we have $L\vec{y}(i,j) = p\vec{y}(i,j)$. The $\vec{y}$'s span a $(q-1)$-dimensional subspace for eigenvalue p. We are missing one eigenvalue; call it λ. The easiest way to find λ is probably to note that $Tr(L) = 2pq$ (the sum of the degrees of the vertices, so twice the number of edges), so $p(q-1) + q(p-1) + \lambda = 2pq$. We deduce that $\lambda = p + q$. The actual eigenvector is $(q, q, \ldots, q, -p, -p, \ldots, -p)$.

In summary, we have confirmed the formula $\det(x+L) = x(x+q)^{p-1}(x+p)^{q-1}(x+p+q)$.

(c) How many spanning trees does $K_{p,q}$ have?

Using the formula from class, the number of spanning trees is $\frac{1}{p+q}q^{p-1}p^{q-1}(p+1) = q^{p-1}p^{q-1}$.

Problem 2 *Let G_d be the graph which has $2d$ vertices, $u_1, \ldots, u_d, v_1, \ldots, v_d$ and where every vertex is connected to every other vertex* except *that there is no edge from u_i to v_i.*

Let S be the $2d \times 2d$ matrix with a 1 in positions $(i, i+d)$ and $(i+d, i)$, and zeroes everywhere else. Let J be the $2d \times 2d$ matrix whose every entry is a 1.

(a) Express the Laplacian matrix of G_d in terms of S, J and Id.

We have $L = (2d-1)\mathrm{Id} - J + S$.

(b) Check that $S^2 = \mathrm{Id}$, $SJ = JS = J$ and $J^2 = 2dJ$. What are the possible eigenvalues of S and J?

The equations are easy to check. From the equation $S^2 = \mathrm{Id}$, the eigenvalues of S are ± 1. From the equation $J^2 = 2dJ$, or from computations in class, the eigenvalues of J are 0 and $2d$.

(c) Show that S and J can be simultaneously diagonalized. Describe the corresponding eigenspaces and eigenvalues.

Let V_+ and V_- be the eigenspaces of S. Since $SJ = JS$, we see that $JV_\pm = V_\pm$. Define inner products on V_+ and V_- by restricting the standard inner product from $\mathbb{R}^{2d}$. Since J is symmetric, J restricted to each of these subspaces is self-adjoint and, thus, $J|_{V_+}$ and $J|_{V_-}$ are diagonalizable.

Explicitly, we have the following eigenspaces: The span of $(1, 1, \ldots, 1)$, with eigenvalues $(2d, 1)$ for (J, S). The d dimensional space of functions whose value at u_i is negative the value at v_i, with eigenvalues $(0, -1)$ for (J, S). And the $(d-1)$-dimensional space of functions which take the same value at u_i and v_i, and sum to 0, with eigenvalues $(0, 1)$.

(d) How many spanning trees does G_d have?

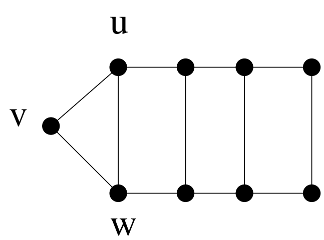
On the above listed subspaces, L acts by $(2d-1) - (2d) + 1 = 0$, $(2d-1) - 0 + (-1)$ and $(2d-1) - 0 + 1$ respectively. So the characteristic polynomial is

$$x(x+2d-2)^d(x+2d)^{d-1}.$$

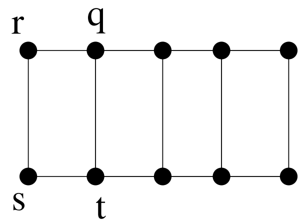
There are $\frac{1}{2d}(2d-2)^d(2d)^{d-1} = (2d-2)^d(2d)^{d-2}$ spanning trees.

Problem 3 *Let G_n and H_n be the graphs shown on the problem set. Let $\tau(G)$ denote the number of spanning trees of the graph G. Prove that*

$$\begin{aligned}\tau(G_n) &= \tau(G_{n-1}) + 2\tau(H_{n-1}) \\ \tau(H_n) &= \tau(G_{n-1}) + 3\tau(H_{n-1})\end{aligned}$$



In the graph G_n, consider the vertices u, v and w marked above. Any spanning tree must use at least 1 of the edges uv and vw. Suppose the tree only uses one of these edges. Delete that edge, and you obtain a spanning tree of H_{n-1}.



On the other hand, suppose that we use both uv and vw. Deleting these edges gives a spanning forest of H_{n-1} where r and s are in separate components. Contracting the edge rs gives a spanning tree of G_{n-1}. Altogether, we have $\tau(G_n) = 2\tau(H_{n-1}) + \tau(G_{n-1})$.

The other equation can be proved similarly: The $3\tau(H_{n-1})$ term corresponds to using two of the edges qr, rs and st; the $\tau(G_{n-1})$ term corresponds to using all three.

Problem 4 *Let T_n be the number of trees on vertex set $\{1, 2, \ldots, n\}$. (We will prove in class that $T_n = n^{n-2}$, but it is easiest to do this problem without using that fact.)*

Prove that

$$2(n-1)T_n = \sum_{k=1}^{n-1} \binom{n}{k} kT_k(n-k)T_{n-k}.$$

We will show that both sides of the equation count triples (T, u, v) where T is a tree on the vertex set $\{1, 2, \ldots, n\}$ and u and v are adjacent vertices. The left side is easy: T_n is the ways to choose the tree T; the term $(n-1)$ is the number of ways to choose the edge joining u and v; and the factor of 2 counts choosing which is u and which is v.

On the right hand side, removing the edge (u, v) divides T into two trees U and V, where u contains U and v contains V. The k-th term on the right hand side is the number of triples (T, u, v) where U has k vertices and V has $n-k$. Namely, we first choose which vertices to put into U and which to put into V; this can be done in $\binom{n}{k}$ ways. We then choose a tree structure on the k-element set, in T_k ways, and on the $n-k$ element set in T_{n-k} ways. Finally, we choose which vertex of each tree to connect to the other tree; giving $k(n-k)$ choices.