Problem 1 *Let G be a graph with n vertices and e edges, so that every edge lies in m spanning trees. Consider the graph as made of identical resistors with resistance R. What is the effective resistance between two adjacent vertices?*

Let u and v be two adjacent vertices. From class, the effective resistance between u and v is

$$R\frac{\#(\text{two tree forests separating } u \text{ and } v)}{\#(\text{spanning forests})}.$$

The numerator is the same as the number of spanning trees of G which contain the edge from u to v. So the effective resistance is the probability that a random tree will contain edge e.

Let t be the total number of spanning trees. Since each tree has $n-1$ edges, and each edge is in m trees, we have $t(n-1) = em$. We want to compute $m/t = (n-1)/e$. So the final answer is $(n-1)/e \times R$.

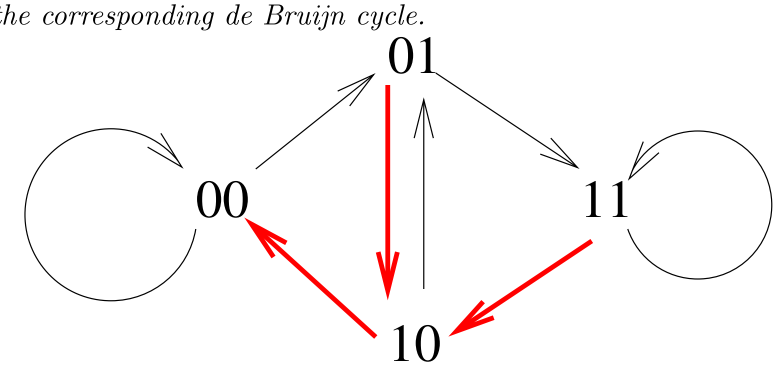
Problem 2 *Let n be a positive integer. Construct a binary string in a greedy manner as follows: Start with $n-1$ copies of 0. Add bits to the end as follows: Add 1 unless this would cause the same n-bit substring to occur twice. If we are forbidden to add 1, then add 0 if this will not cause an n-bit substring to reoccur. If both 1 and 0 would cause an n-bit substring to reoccur, then stop. For $n = 3$, this produces*

$$0011101000.$$

In this problem, we will show that this procedure ends with $n-1$ zeroes and that gluing those zeroes to the initial zeroes produces a de Bruijn cycle.

Let D_n be the directed graph whose vertices are binary strings of length $n-1$ and where there is an edge from $b_1b_2\cdots b_{n-1}$ to $b_2b_3\cdots b_{n-1}b_n$. (This is the graph we used in class to prove de Bruijn sequences exist.) Let T be the subgraph consisting of the edges $b_1b_2\cdots b_{n-1} \longrightarrow b_2\cdots b_{n-1}0$ for every binary string $b_1b_2\cdots b_{n-1}$ other than $00\cdots 0$.

(a) For $n = 3$, draw the graph D_3 and the subgraph T. You should see that T is a rooted subtree of D_3; construct the corresponding de Bruijn cycle.



(b) Prove in general that T is a rooted subtree of D_n.

We will show that, for every vertex v of T, there is a directed path from v to $00\ldots0$. This shows that there cannot be any directed cycles, so T is rooted at $00\ldots0$. This is simple: Let $v = v_1v_2\cdots v_{n-1}$. Then

$$(v_1v_2\cdots v_{n-1}) \longrightarrow (v_2\cdots v_{n-1}0) \longrightarrow (v_3\cdots v_{n-1}00) \longrightarrow \cdots \longrightarrow (v_{n-1}0\cdots 0) \longrightarrow (00\cdots 0)$$

is a directed path in T.

(c) Prove that the Eulerian walk of D_n corresponding to T (by the BEST algorithm) gives the de Bruijn sequence constructed by the greedy procedure above.

In the BEST algorithm, we first leave a given vertex using the edge not in T and then, the next time, we take the edge from T. (If the out-degree were greater than 2, we would also need a way to choose between the various out-going edges not in T, but the out-degree is 2.) The edge of T

pointing out of $v_1 v_2 \cdots v_{n-1}$ is to $v_1 \cdots v_{n-1} 0$, so this algorithm says to choose 1 before choosing 0, precisely as the greedy algorithm specifies.

Problem 3 *Let G be a directed graph with n vertices v_1, v_2, ..., v_n arranged around a circle. G has $4n$ edges: For each vertex, there are two edges to its clockwise neighbor and two to its counter-clockwise neighbor.*

(a) Find the number of rooted spanning trees of G. (Just thinking about the problem is probably easier than using the matrix-tree theorem.)

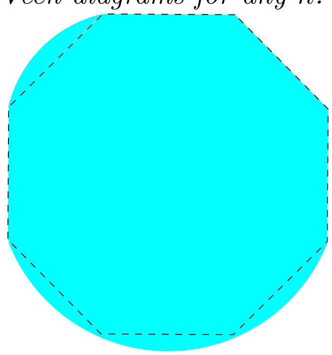
We must choose precisely one of the pairs (v_i, v_{i+1}) not to be joined by an edge. We must choose exactly one of the vertices to be the root. Finally, for each of the $n-1$ pairs (v_j, v_{j+1}) which are joined by an edge, we have 2 choices of which edge to use. This gives $n^2 2^{n-1}$ trees total. If we specify a particular root, there are $n2^{n-1}$.

(b) Find the number of Eulerian walks in G.

The number of Eulerian walks starting (and hence ending) at v_i will be $\left(n2^{n-1}\right) \times 4! \times (3!)^{n-1}$. So the total number is $n^2 \times (12)^{n-1} \times 24$.

Problem 4 *Let n be a positive integer. Let X_1, X_2, ..., X_n be subsets of the plane $\mathbb{R}^2$. For any subset I of $\{1, 2, \ldots, n\}$, let $Y_I = \left(\bigcap_{i \in I} X_i\right) \cap \left(\bigcap_{j \notin I} X_j^C\right)$, where X_j^C is $\mathbb{R}^2 \setminus X_j$. We will define X_1, ..., X_n to be a generalized Venn diagram if all the X_i's are convex and all the Y_I's are nonempty.*

Show that there exist generalized Veen diagrams for any n.



Let P be a regular 2^n inscribed in a circle. Label the 2^n wedges between P and the circle with a binary de Bruijn cycle. Let X_1 be the union of P and the wedges which are labeled 1. Let X_k, for $2 \leq k \leq n$, be the rotation of X_k by $2\pi k/2^n$. As can be seen from the figure, X_k is convex. For I other than $\emptyset$ or $\{1, 2, \ldots, n\}$, the intersection Y_I is a single wedge. $Y_\emptyset$ is the exterior of the circle, plus one more wedge. Finally, $Y_{\{1,2,\ldots,n\}}$ is the union of P and one wedge.