Solution Set 5

Problem 1 *Let G be a finite graph and let A be its adjacency matrix. Let d be the largest degree of any vertex of G.*

(a) Show that the eigenvalues of A lie in the interval $[-d, d]$.

Let d_i be the degree of vertex v_i.

We must show that $A + d\mathrm{Id}$ is positive definite and $A - d\mathrm{Id}$ is negative definite. For the former, consider any vector $x = (x_1, x_2, \dots, x_n)$. Then

$$x^T(A + d\mathrm{Id})x = d\sum x_i^2 + 2 \sum_{(i,j) \text{ an edge}} x_i x_j = \sum (d - d_i)x_i^2 + \sum_{(i,j) \text{ an edge}} (x_i + x_j)^2 \geq 0.$$

Similarly,

$$x^T(A - d\mathrm{Id})x = -d\sum x_i^2 + 2 \sum_{(i,j) \text{ an edge}} x_i x_j = -\sum (d - d_i)x_i^2 - \sum_{(i,j) \text{ an edge}} (x_i - x_j)^2 \leq 0.$$

(b) Give a simple criterion for when d is an eigenvalue of A.

We want to know when there is a nontrivial x in the kernel of $A - d\mathrm{Id}$. From the above, this happens if and only if we can arrange for $\sum (d_i - d)x_i^2 - \sum_{(i,j) \text{ an edge}} (x_i - x_j)^2$ to be zero. From the second sum, we see that such an x must be constant on every connected component of G. From the first terms, we see that we must have $d_i = d$ whenever x_i is nonzero. Combining these, we see that d is an eigenvalue of the adjacency matrix if and only if G **has a connected component in which every vertex has degree d.**

(c) Give a simple criterion for when $-d$ is an eigenvalue of A.

As in the last problem, we want $\sum (d - d_i)x_i^2 + \sum_{(i,j) \text{ an edge}} (x_i + x_j)^2$ to be zero. This means that $d = d_i$ whenever x_i is nonzero and, if x_i and x_j are joined by an edge, then $x_i = -x_j$. In other words, $-d$ is an eigenvalue of the adjacency matrix if and only if G **has a connected bipartite component where every vertex has degree** d.

Problem 2 *Let G be the Petersen graph and A it's adjacency matrix. We showed in class that A has eigenvalues $3, 1, 1, 1, 1, 1, -2, -2, -2, -2$.*

Let v be a vertex of G. Give a formula for the number of walks G of length k, starting and ending at v. Hint: For $k = 1$, 2 and 3, you should get 0, 3 and 0.

The number of closed walks of length k is $\mathrm{Tr}(A^k) = 3^k + 5 + 4 \times (-2)^k$. Since the symmetry group of the Petersen graph can take any vertex to any other, we divide by 10 to have our walk start at a particular vertex:

$$\frac{1}{10}\left(3^k + 4 \times (-2)^k + 5\right).$$

Problem 3 *Let G be a finite graph with adjacency matrix A and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.*

(a) Show that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ is a nonnegative integer and give a combinatorial description of it.

This is the trace of A^2, so it is $\sum_{v \in G} \deg(v)$, or twice the number of edges of G.

(b) Show that $\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_n^3$ is a nonnegative integer and give a combinatorial description of it.

This is the trace of A^3, so it is the number of length 3 walks in G, or 6 times the number of triangles.

(c) Suppose that G has n vertices and every vertex has degree d. Show that

$$\lambda_1^4 + \lambda_2^4 + \cdots + \lambda_n^4 - (nd + nd(d-1) + nd(d-1))$$

is a nonnegative integer and give a combinatorial description of it. (The expression in parentheses is deliberately not simplified as a hint.)

The sum $\lambda_1^4 + \lambda_2^4 + \cdots + \lambda_n^4$ is the number of closed walks of length 4. The other terms count degenerate walks of various sorts: The quantity dn is the number of walks of the form $v_i \to v_j \to v_i \to v_j \to v_i$; the quantity $nd(d-1)$ is the number of walks of the form $v_i \to v_j \to v_k \to v_j \to v_i$ with $i \neq k$; the quantity $nd(d-1)$ is the number of walks of the form $v_j \to v_i \to v_j \to v_k \to v_i$ with $i \neq k$.

After subtracting these out, we are left with the number of closed walks of length 4 using 4 distinct vertices. Alternatively, 8 times the number of 4-cycles in G.

Problem 4(a) *Let G be a finite graph. Let w_k be the number of closed walks in G of length k and let p be a prime number. Show that w_p is divisible by p.*

Group together walks which only differ by shifting their starting point, so $v_1 \to v_2 \to v_3 \to \cdots \to v_p \to v_1$, $v_2 \to v_3 \to \cdots \to v_p \to v_1 \to v_2$ and so forth are grouped together. Since p is prime, and G has no self loops, all these equivalence classes have size p.

(b) Give an example of a finite graph for which w_4 is not divisible by 4.

If we group walks that are equivalent up to shift, we get lots of groups of size 4, plus some groups of size 2. Specifically, for each edge (u, v), we get a group of size 2 containing $u \to v \to u \to v \to u$ and $v \to u \to v \to u \to v$. So any graph with an odd number of edges is an example.

(c) Let A be an $n \times n$ integer matrix and let p be a prime. Show that

$$\mathrm{Tr}(A^p) \equiv \mathrm{Tr}(A) \bmod p.$$

You will want Fermat's Little Theorem, which says that $a^p \equiv a \bmod p$ for any integer a. (Hint: Thinking about walks in graphs helps.)

As above, $\mathrm{Tr}(A^p)$ is the sum of $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{p-1} i_p} a_{i_p i_1}$ where the sum is over all closed walks $i_1 \to i_2 \to \cdots i_p \to i_1$ in the complete graph. Grouping together walks which all cycles of each other, the only terms which are not $0 \bmod p$ are the walks which stay at the same vertex the entire time. So we get

$$\mathrm{Tr}(A^p) = \sum_i a_{ii}^p.$$

By Fermat's Little Theorem, the right hand side is $\sum_i a_{ii} = \mathrm{Tr}(A)$.

(d) Find an integer matrix A and a prime p such that $A^p \not\equiv A \bmod p$.

There are tons of examples; $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ works.

Problem 5 *Let G be a finite graph with at least two vertices such that, for any two distinct vertices u and v, there are exactly two paths of length 2 from u to v.*

(a) Show that all vertices of G have the same degree.

Let x and y be two vertices of G with degrees d and e. First, suppose that x and y do not border each other. Let $x_1, x_2, \ldots, x_d$ be the neighbors of x and let $y_1, y_2, \ldots, y_e$ be the neighbors of y. Then, for every x_i, there are two paths of length two from x_i to y, so x_i borders two of the y_j. Thus, the total number of edges which connect some x_i to some y_j is $2d$. But, symmetrically, this number is also $2e$, so $2d = 2e$ and $d = e$.

Now suppose that x and y do border each other. Let $x_1, x_2, \ldots, x_{d-1}$ be the other neighbors of x and let $y_1, y_2, \ldots, y_{e-1}$ be the other neighbors of y. There are two paths of length two from x_i to y. One of them is $x_i \to x \to y$. The other must be of the form $x_i \to y_j \to y$ for some y_j. So each x_i borders one y_j. There are thus $d-1$ edges which connect some x_i to some y_j. Symmetrically, there are $e-1$ such edges. So $d-1 = e-1$ and $d = e$.

Let d be this common degree and let n be the number of vertices of A.

(b) Give a formula for n in terms of d.

Fix a starting vertex v and consider all non-backtracking paths of length 2 starting at v. There are $d(d-1)$ of them. For each $w \neq v$, there are two such paths ending at w. So $2(n-1) = d(d-1)$, or $n = d(d-1)/2 + 1$.

(c) By analyzing the spectrum of A, show that there are only two possible values of d.

We have $A^2 = 2J + (d-2)$. The eigenvalues of J are $n = d(d-1)/2 + 1$, with multiplicity 1, and 0, with multiplicity $n-1$. So the eigenvalues of $2J + (d-2)$ are d^2, with multiplicity 1, and $d-2$, with multiplicity $n-1$.

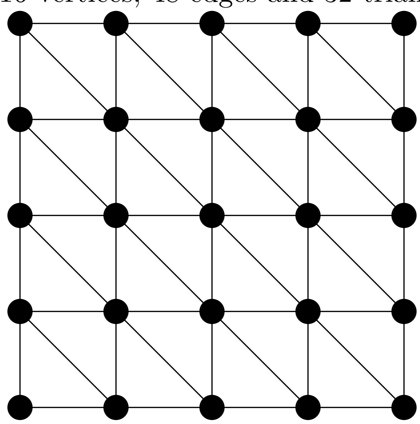
So the eigenvalues of A are d (with multiplicity 1) and $\pm\sqrt{d-2}$. Let $k_\pm$ be the number of times that $\pm\sqrt{d-2}$ occurs. So $d + k_+\sqrt{d-2} - k_-\sqrt{d-2} = 0$. We deduce that $\sqrt{d-2}$ is rational, say $d = m^2+2$. Then $m^2 + 2 + (k_+ - k_-)m = 0$. So m divides 2 and $m = 1$ or 2. Correspondingly, $d = 3$ or $d = 6$ and $n = 4$ or $n = 16$.

(d) Construct graphs achieving each of the values of d you found in the previous part.

For $(d, n) = (3, 4)$, we must take the complete graph K_4.

For $(d, n) = (4, 16)$, there are two solutions. One is the graph whose vertices are ordered pairs (x, y), with x and $y \in \{1, 2, 3, 4\}$, and an edge between (x, y) and (x', y') if either $x = x'$ or $y = y'$.

The other can be obtained by taking the triangular grid shown below, and gluing the sides of the square to form a torus with 16 vertices, 48 edges and 32 triangles.



I have a case by case proof that these are the only options, but it is too ugly to write out; I'll be curious to see if some of you produce nicer ones.

Based on a problem from the 2012 Moscow Mathematical Olympiad.