Solution Set 6

Problem 1 *Let G be a graph and let $L(G)$ be the adjacency matrix of G. Let $\lambda_2(G)$ be the second smallest eigenvalue of $L(G)$. (The smallest eigenvalue is 0, corresponding to the all 1 vector.)*

(a) Let G' be a graph formed by adding an extra edge to G. Show that $\lambda_2(G') \geq \lambda_2(G)$.

Let the extra edge of G' join i and j. Let v_2' be the eigenvector of G'. We have $\sum_i v_2'(i) = 0$ so we may plug into the Raleigh quotient:

$$\lambda_2(G) \leq \frac{(v_2')^T L(G) v_2'}{(v_2')^T v_2'} = \frac{\sum_{(k,\ell)\in \mathrm{Edge}(G)} (v_2'(k) - v_2'(\ell))^2}{\sum_k (v_2'(k))^2} =$$

$$\frac{\sum_{(k,\ell)\in \mathrm{Edge}(G')} (v_2'(k) - v_2'(\ell))^2 - (v_2'(k) - v_2'(\ell))^2}{\sum_k (v_2'(k))^2} \leq$$

$$\frac{\sum_{(k,\ell)\in \mathrm{Edge}(G')} (v_2'(k) - v_2'(\ell))^2}{\sum_k (v_2'(k))^2} = \frac{(v_2')^T L(G') v_2'}{(v_2')^T v_2'} = \lambda_2(G')$$

Problem 2 *Let H be a bipartite graph and let $A(H)$ be its adjacency matrix. Show that, if λ is an eigenvalue of H, then $-\lambda$ is also an eigenvalue of H.*

Let B be the set of black vertices of H and let W be the set of white vertices. Let v be an eigenvector of H; write v as $b+w$ where b is supported on B and w is supported on W. Then $b-w$ is an eigenvector with eigenvalue $-\lambda$.

Problem 3 *Let G be a d-regular graph on n vertices. Let $A(G)$ be the adjacency matrix of G, with eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

(a) Show that $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = dn$.

Since $\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$ is $Tr(A^2)$, it is the number of paths $u \to v \to u$ in A. This is twice the number of edges, so i is the sum of the degrees of the vertices, which is dn.

(b) Show that $\max(|\lambda_2|, |\lambda_n|) \geq \sqrt{\frac{dn-d^2}{n-1}}$.

Let $g = \max(|\lambda_2|, |\lambda_n|)$. Then $d^2 + (n-1)g^2 \geq \sum \lambda_i^2 = dn$ and some algebra gives $g \geq \sqrt{(dn-d^2)/n-1)}$.

(c) Fix d and fix $\epsilon > 0$. Show that, for n sufficiently large, $\max(|\lambda_2|, |\lambda_n|)$ is $> \sqrt{d} - \epsilon$.

As $n \to \infty$, we have $\sqrt{(dn-d^2)/n-1)} \to \sqrt{d}$.

Problem 4.(a) *Let G be the n-cycle and let $L(G)$ be the Laplacian matrix of G. Show that the eigenvalues of $L(G)$ are $2 - 2\cos(2\pi j/n)$, for $0 \leq j < n$.*

For each j, the vector $k \mapsto \exp(2\pi i j k/n)$ is an eigenvector of the Laplacian, with eigenvalues $2 - \exp(2\pi i j k/n) - \exp(-2\pi i j k/n)$.

(b) Let H be the path of length n and let $L(H)$ be the Laplacian matrix of H. Let G be the $2n$-cycle with Laplacian matrix $L(G)$. Show that every eigenvalue of $L(H)$ is also an eigenvalue of $L(G)$. More specifically, show that the eigenvalues of H are $2 - 2\cos(2\pi j/(2n))$, for $0 \leq j < n$.

Let $(v_1, v_2, \ldots, v_n)$ be a function on the vertices of H. Define the function $\phi(v)$ on the vertices of G to be $(v_1, v_2, \ldots, v_n, v_n, \ldots, v_2, v_1)$. Then $L(G)\phi(v) = \phi(L(H)v)$. In particular, if v is an eigenvector of H with eigenvalue λ, then $\phi(v)$ is an eigenvector of $L(G)$ with eigenvalue λ. In particular, every eigenvalue of $L(H)$ is also an eigenvalue of $L(G)$. So the eigenvalues of $L(H)$ are a subset of $2 - 2\cos(2\pi j/(2n))$.

To check that each value of $2 - 2\cos(2\pi j/(2n))$ occurs, probably the easiest method is to write down an explicit eigenvector. The corresponding eigenvector is $k \mapsto \cos(2\pi j(k - 1/2))/(2n)$

(c) Let S be the tree with $n+1$ vertices, one of which borders all the others. Find the eigenvalues of $L(S)$.

Some experimentation will quickly convince you that 1 occurs with multiplicity $n-1$, and the other eigenvalues are 0 and $n+1$. The eigenspace for 1 consists of all those vectors which have a 0 at the center and where the values on the edges add up to 0. The eigenspace for 0, of course, is the all 1's vector.

We can find the last eigenvalue by recalling that he eigenvectors must add up to $Tr(L(S)) = 2n$. Alternatively, since eigenvectors for different eigenvalues are orthogonal, we can compute the last eigenvector as the orthogonal complement to the ones we already know of: The vector which is 1 on all the leaves and $-n$ at the center.