PROBLEM SET 7 – DUE NOVEMBER 5TH

Please see the course website for homework policy.

***Problem 1*** *Fix a positive integer d; in this problem we will study d-regular graphs on n vertices. Let $A(G)$ be the adjacency matrix and $\lambda_i$ its eigenvalues. On the previous problem set, we proved the following: For any $\epsilon > 0$, there is an $N$ such that, if $n > N$, then $\max(|\lambda_2|, |\lambda_n|) > \sqrt{d} - \epsilon$. In this problem, we will improve $\sqrt{d}$ to $\sqrt{2d-1}$. Fix $c > 0$.*

*(a) Let $B = A^2 - (d-(1+c)/2)$. Show that $Tr(B^2) \geq n(d^2 - d + (1+c)^2/4)$. When does equality occur?*

For any square matrix $B$, we have $Tr(B^2) = \sum_{i,j} B_{ij}B_{ji}$. The diagonal elements of $B$ are each $d-(d-(1+c)/2) = (1+c)/2$, contributing $n(1+c)^2/4$. If the graph $G$ contains no quadrilaterals, then the off diagonal entries of each row of $A$ contain $d^2 - d$ ones and all the others are zeroes, so these contribute $n(d^2 - d)$ and we have equality. In general, the off diagonal entries of each row of $A$ are nonnegative integers summing to $d^2 - d$. If $x_i$ are nonnegative integers, then $\sum x_i^2 \geq \sum x_i$, with equality if and only if each $x_i$ is 0 or 1. So we obtain the bound, and we have equality exactly when none of the off diagonal entries are greater than 1, which is when there are no quadrilaterals.

Let $g = \sqrt{2d-1-c}$.

*(b) Suppose that all the eigenvalues of A, other than d, are in $(-g, g)$. Show that $Tr(B^2) \leq (d^2 - d + (1+c)/2)^2 + (n-1)(d-(1+c)/2)^2$.*

Let the eigenvalues of $A$ be $\lambda_i$. So $Tr(B^2) = \sum(\lambda_i^2 - (d-(1+c)/2))^2$. The $\lambda_1 = d$ term contributes $(d^2 - (d-(1+c)/2))^2$. When $\lambda$ is in $(-g, g)$, the quantity $(\lambda^2 - (d-(1+c)/2))$ is between $-(d-(1+c)/2)$ and $(d-(1+c)/2)$ so those terms contribute at most $(d-(1+c)/2)^2$.

*(c) Show that the above inequalities imply a finite upper bound for n (dependent on c and d).*

Combining the two inequalities above, we have

$$n(d^2 - d + (1+c)^2/4) \leq (d^2 - d + (1+c)/2)^2 + (n-1)(d-(1+c)/2)^2.$$

Algebraic rearrangement gives:

$$n \leq \frac{d((d-1)^2 + c)}{c}.$$

***Problem 2*** *Let G be a graph. We define a double of G to be a graph DG as follows: Each vertex v in G gives two distinct vertices $v_1$ and $v_2$ in DG. If there is an edge $(v, w)$ in G, then* either *there are edges $(v_1, w_1)$ and $(v_2, w_2)$, or else there are edges $(v_1, w_2)$ and $(v_2, w_1)$ (but not both). There are no other edges in DG.*

*(a) Show that every eigenvalue of $A(G)$ is an eigenvalue of $A(DG)$.*

Let $x$ be an eigenvector of $A(G)$ with eigenvalue $\lambda$. Define $Dx$ to be the vector with $Dx(v_1) = Dx(v_2) = x(v)$ for every vertex $v$ in $G$. Then $A(DG)(Dx) = \lambda Dx$.

*(b) Define a matrix B as follows: there is a row and a column of B for every vertex of G. For two vertices v and w of G, we have $B_{vw} = 0$ if there is no edge $(v, w)$ in G; we have $B_{vw} = 1$ if edges $(v_1, w_1)$ and $(v_2, w_2)$ occur in DG; and we have $B_{vw} = -1$ if edges $(v_1, w_2)$ and $(v_2, w_1)$ occur in DG. Show that every eigenvalue of B is an eigenvalue of $A(DG)$.*

Let $W$ be the vector space of functions on $DG$ which obey $x(v_1) = -x(v_2)$. Then $A(DG)$ takes $W$ to itself and the matrix $B$ gives the action of $A(DG)$ on $W$.

***Problem 3*** *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \cdots \lambda_n$. Let B be the upper left $m \times m$ submatrix, with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$.*

*(a) Show that $\mu_1 \leq \lambda_1$.*

Let $w_1$ be the eigenvector of $B$ corresponding to $\mu_1$. Then

$$\frac{w_1^T A w_1}{w_1^T w_1} = \mu_1$$

so, as $\lambda_1$ is the maximal value of the Raleigh quotient, we have $\lambda_1 \geq \mu_1$.

*(b) Let $\vec{v}_i$ be the eigenvectors of $A$. Show that there is some vector $\vec{u}$ which is supported in the first $m$-coordinates and is in the span of $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{n-m}$, $\vec{v}_{1+n-m}$. Use this fact to show that $\mu_1 \geq \lambda_{1+n-m}$.*

Following the hint, the span of $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_{n-m}$, $\vec{v}_{1+n-m}$ has dimension $n-m+1$ The space of vectors supported on the first $m$ coordinates has dimension $m$. So these two spaces have nonzero intersection; let $u$ be in this intersection. Then

$$\frac{u^T B u}{u^T u} = \frac{u^T A u}{u^T u} \geq \lambda_{n-m+1}$$

Since $\mu_1$ is the largest value of the Raleigh quotient for $B$, we have $\mu_1 \geq \lambda_{n-m+1}$.

*(c) If we repeat these arguments for $\mu_m$, what bounds do we get?*

$$\lambda_m \geq \mu_m \geq \lambda_n.$$

*(d) Write $C$ for the upper left $(m-k+1) \times (m-k+1)$ submatrix and $\gamma_i$ for its eigenvalues. Show that we can change bases, without changing the eigenvalues of $A$ and $B$, so that $\mu_k = \gamma_1$. Deduce that $\mu_k \geq \lambda_{k+n-m}$.*

Let the matrix $S$ diagonalize $B$. More specifically, we want $SBS^{-1} = \mathrm{diag}(\mu_k, \mu_{k+1}, \ldots, \mu_m, \mu_1, \ldots, \mu_{k-1})$. So $\left(\begin{smallmatrix} S & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right) A \left(\begin{smallmatrix} S & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right)^{-1}$ has upper left block $\mathrm{diag}(\mu_k, \mu_{k+1}, \ldots, \mu_m, \mu_1, \ldots, \mu_{k-1})$. We have not changed the eigenvalues of $A$ or of the upper left block of $B$.

Now, $\mu_k = \gamma_1$. Applying part (b) to the matrices $C$ and $A$, we have $\gamma_1 \geq \lambda_{n-(m-k+1)+1}$ so $\mu_k \geq \lambda_{k+n-m}$.

*(e) Write $D$ for the upper left $k \times k$ submatrix and $\delta_i$ for its eigenvalues. Show that we can change bases, without changing the eigenvalues of $A$ and $B$, so that $\mu_k = \delta_k$. Deduce that $\mu_k \leq \lambda_k$.*

We repeat the previous argument, this time taking $T$ to diagonalize $B$ with $TBT^{-1} = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_m)$ and considering $\left(\begin{smallmatrix} T & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right) A \left(\begin{smallmatrix} T & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right)^{-1}$. Using the lower bound from (c),

$$\delta_k \geq \lambda_k \quad \text{and so} \quad \mu_k \geq \lambda_k.$$

*(f) The above results give some very crude restrictions on subgraphs of Ramanujan graphs. For example, suppose that $G$ is a 10 regular graph with all eigenvalues of the adjacency matrix other than $\lambda_1$ in $[-6, 6]$. Show that $G$ does not contain two seven element subsets $X$ and $Y$ so that there are edges joining every $x \in X$ to every $y \in Y$, and so that there are no edges within $X$ or within $Y$.*

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